**ENTROPY AND COMPLEXITY IN URBAN AND REGIONAL SYSTEMS**

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July, 2019

Abstract:

This paper considers the roles of the concepts of entropy and complexity in the modeling of urban and regional systems over time. Drawing on Rosser (2016), the concept of entropy will be reviewed from its nineteenth century as the Second Law of Thermodynamics development for statistical mechanics applications through the twentieth century information entropy to more generalized forms. The seminal application to urban and regional economics is due to Wilson (1970) who showed how minimizing transport costs in a market flow model obeys an entropic formulation. Besides the this influential model, entropy has been used in studies of spatial settlement distribution, spatial concentration and dispersion, urban sprawl, racial segregation, land use diversity, and measures of sustainability of energy use in urban and regional systems. These entropic forces compete with anti-entropic tendencies associated with agglomeration economies that are associated with the emergence of a power law distribution of the urban hierarchy, with these possibly following Zipf rank-size rule, a matter of long debate. The interaction of these dynamic processes underlies complex dynamics that generate a self-organizing pattern of urban and regional development.

Keywords: entropy, complexity, urban, regional, spatial, dynamical systems

“…classical thermodynamics is…the only physical theory of universal content which I am convinced, within the framework of the framework of applicability of its basic concepts will never be overthrown.” - Albert Einstein, quoted in Rifkin (1981, p. 44)

1. **Introduction**

Since at least the early efforts of Alan Wilson (1967, 1970), the idea of using the law of entropy to assist in modeling the development of urban and regional spatial structural patterns has been influential. To understand how this has been done and how useful it is as an approach, we must first consider the various formulations of that law that have been made. The full development of the idea is associated with the second law of thermodynamics due principally to Boltzmann (1884), although drawing on earlier work by Carnot (1824) and Clausius (1865). Jaynes (1957) prepared this approach for application in economics with Georgescu-Roegen (1971) also providing a deep perspective. Later, Shannon (1948) would extend this to the study of information patterns. Rosser (2016) argues that within economic systems the former is most appropriate when ontological thermodynamic forces are objectively driving the dynamics of a system. The latter is more important as a metaphorical tool when a similar mathematical pattern arises.[[1]](#footnote-1)

A way in which the first may generate structural patterns is through the operation of energy in the system, given that the second law of thermodynamics is about how energy dissipates through closed systems. Energy is crucial in transportation, so it should not be surprising that as transportation costs enter into determining such spatial patterns we might see the law of entropy in its objective form as relevant to shaping such patterns, and indeed, transportation costs have been seen as central in shaping urban and regional spatial patterns dating back to von Thünen (1826). Drawing on a proposal by Reilly (1931) and work by Weaver (1948). Wilson (1967, 1870, 2010) would use the assumption of minimizing transportation costs to model a complex system of spatial distribution of rent-maximizing activities. Another early effort along similar lines was due to Medvedkov (1967).

Many applications of entropy or urban and regional models would follow the metaphorical approach based on the Shannon’s (1948) information entropy. An early effort along these lines was due to Chapman (1970) for a model of spatial concentration or dispersion of activities and also Batty (1976). Likewise this has underpinned models of urban sprawl (Cabral et al., 2013). Indexes of degrees of racial segregation have been based on such measures (Mora and Ruiz-Castillo, 2011). Likewise, measures for land-use diversity have been based on such entropy (Walsh and Webber, 1977).

Rather returning to fundamental thermodynamic formulation have been efforts to model ecological sustainability of urban and regional systems based on their patterns of energy usage. Assessing carbon footprints is due to Wackernagel and Rees (1996). More direct applications including using the concept of exergy are due to Balocco et al. (2000). Marchinetti et al. (2006) consider such models under within the complex systems dynamics of dissipative systems (Prigogine 1980).

An alternative stresses anti-entropic forces associated with agglomeration for modeling patterns of urban hierarchy[[2]](#footnote-2) reflecting power law distributions initiated by Pareto (1897), supported by Singer (1936) and Gabaix (1999). A special case is the rank-size rule due to Auerbach (1913) and Zipf (1949), supported by Batten (2001), Nitsch (2005), and Berry and Okulicz-Kozaryn (2012).

Finally models of complex self-organization of urban and regional structure reflecting interactions between entropic and anti-entropic elements have been developed by many including Papageorgiu and Smith (1983), Weidlich and Haag (1987, Krugman (1996), Portugali (1999), Gabaix and Ioannides, 2004), and Rosser (2011). These interactions can trigger the irregularities in dynamic paths that mark dynamically complex systems, which urban and regional systems surely are.

1. **The Law of Entropy**

While the term has had many definitions since Rudolf Clausius (1865) coined the term *entropy* from the Greek for “transformations,” Ludwig Boltzmann (1896) identified it as the state of disorder (or randomness) in a system. The law of entropy, or second law of thermodynamics, thus becomes that in a closed system entropy increases, which was first formulated by Clausius (1865), who also stated the classical fist law of thermodynamics that in a closed system the amount of energy is constant, with this more fully developed by Ludwig Boltzmann (1884).

The inspiration for this development came from the study of steam engines by Sadi Carnot (1824). He made the initial crucial observation of the first law, which would be crucial to understanding the impossibility of a perpetual motion machine. Carnot formulated that the work of a steam engine came from the transformation of heat energy from a hotter source to a cooler sink and recognized a maximum efficiency for this transformation.[[3]](#footnote-3) It was from this understanding that Clausius derived his conceptualization, later adumbrated by Boltzmann.

The version of the most general form of entropy was never written down by Boltzmann but that appears on his grave is

*S = k*ln *W*, (1)

Where *S*is entropy, *k* is the Boltzmann constant, and *W* is the thermodynamic probability of an aggregate state of a system of gas molecules that Boltzmann (1896) formulated as being given by

*W*= *N*!/Π*n*i!, (2)

where *N* is the number of microscopic states in the system, with the probability of a gas molecule being in the ith state is *n*i/*N*.[[4]](#footnote-4)

The most important variation of this would the metaphorical one measuring informational entropy due to Shannon (1948) and Shannon and Weaver (1949). This measures the probability distribution of states informational uncertainty, *H(pi)*  for message i, given by

*H*(*p1*…*pn*) = -*k*Σ*pi*ln*pi*. (3)

While these two forms of entropy apply to very different situations with no ontological law of entropy operating with regard to Shannon’s metaphorical informational entropy, they are fundamentally related.[[5]](#footnote-5) In particular as *N* goes to infinity Shannon entropy will converge on a proportion of Boltzmann entropy.

This fundamental unity extends to later variations and generalizations of the entropy concept as developed by Renyi (1961), Tsallis (1968), and Thurner and Hanel (2012). This latter links to a development in Russia of the “new entropy” that links to ergodicity theory where entropy is seen as an isomorphism between Bernoulli states (Kolmogorov, 1958; Sinai, 1959; Ornstein, 1970).

1. **The Wilson Model**

The most influential modeler of urban and regional systems to use the concept of entropy has been Sir Alan G. Wilson (1967, 1969, 1970, 2000, 2010). His original main model was of the spatial distribution of flows of retail activity, based on a model of Reilly (1931). The space is partitioned by origins *I* and destinations *j* (often a central place) so that *Sij* is a matrix of money flows from origins *I* to retails sites *j*. Then the entropy to be maximized subject to budget constraints of the flows is given by

Max *S* = -Σ *Sij*  ln *Sij*, (4)

where for benefits of a retail site given by *Wj* and costs of going from an origin to a retail site given by *cij* this will give a rent-maximizing spatial distribution

*S* = Σ*Wj* exp (*cij*). (5)

This then could be further modified by specifying more activities with population levels and types of retail outlets. In principle this is broadly consistent with the original von Thünen (1826) model of ring-patterned rent around a central place, although Wilson rarely stressed this point.

This basic model due to Wilson has since gone through many modifications and extensions, including many by Wilson himself with various coauthors. Thus while Wilson originally assumed that transport costs grow linearly with the log of benefits, both may be logarithmic, which might be true for a model of long trips involved in interurban transport, with other functional forms possible as constraints get adjusted accordingly (Haynes and Phillips, 1987).

The model has also been extended to various other applications. Thus Rees and Wilson (1976) and Rogers (2008) placed this into models of migration flows. Straussfogel (1991) used it in studies of suburbanization. In models of trade flows, input-output relations can be introduced into integrated models (Kim et al., 1983; Roy and Flood, 1992).

While the basic model assumed discrete zones, Angel and Hyman (1976) extended entropy-maximizing to continuous space representations. Problems of empirical estimation arise in connection with aggregation and spatial structure in models of spatial interaction (Batty and Skildar, 1982). Econometric models of spatial autocorrelation in this framework have been developed (Berry et al., 2008) as well as broader forms of spatial interaction (Fischer and Griffith, 2008).

Greater emphasis on a metaphoric Shannon information entropy approach was due to Snickars and Weibull (1977). Fotheringham (1983) applied this for the case of competing destination zones. Smith and Hsieh (1997) introduced a Markov equivalent. Anas (1984) links utility maximization and entropy maximization in these models using a multinomial logit model. Wilson (2010) argues that these approaches are consistent with the “disorganized complexity” interpretation of Shannon’s information entropy approach as posited by Weaver (1948). This contrasts with the initial approach of Wilson (1967, 1970) that pursued an entropy approach drawing more on Botlzmann.

A substantial expansion of this framework within the Boltzmann framework was due to Harris and Wilson (1978) who introduced slow dynamics into the model. This took the form of introducing elements derived from Lotka (1925) and Volterra (1938), with Wilson (2008) labeling this result of this combination of Boltzmann, Lotka, and Volterra the “BLV approach.” The slow dynamics allow for growth depending on the profitability of given locations, with the related fast dynamics being shorter term equilibrium adjustment dynamics. This setup provided a basis for considering models of catastrophic bifurcations and cascades (Wilson, 1981; Batty, 2009) as well as chaotic dynamics (May, 1973; Rosser, 1991).[[6]](#footnote-6)

This would eventually lead to a broader consideration of how the Wilson model fits into a broader complexity framework, especially linking with Weaver’s (1948) between organized and disorganized forms of complexity. For this entropy can be seen as providing a key organizing principle (Phillips, 2004; Wilson, 2006) relying on the BLV approach. This has even been proposed to provide an explanation of how entropic based models of lower level flows can provide a foundation for scale free power law distributions of the distributions of settlement area sizes (Dearden and Wilson, 2009), which we shall consider below as associated with anti-entropic organizational principles.

1. **Other Spatial Distribution Entropic Models**

While Wilson’s work inspired a large effort by many people as seen in the previous section, others also used various entropic measures to study spatial distributions in urban and regional systems of various things. One line of research was inspired by applying the Theil (1972) index, which is based on the Shannon information entropy measure. Among the first to do so was Batty (1974). The basic spatial version of the Theil index where *H* is the index, *n* is the number of zones, and *pi* is the probability that variable *x* appears in zone *I*, is given by

*Hn* = [Σ*pi*log(1/*pi*)]/log *n.* (6)

This entropy measure can vary from 0 to 1, with the latter indicating a fully equal distribution across the spatial zones, at maximum entropy, and 0 indicating a total concentration in one zone, or a maximum degree of inequality and anti-entropy. This index has been widely applied across many social and natural sciences.

Batty’s (1974) variation of this, which he called *spatial entropy*, involves considering what happens as the size of the zones shrinks, also implying an increasing number of them. If Δ*xi* is zone size then the Batty spatial entropy index is given by

*H* = (limΔ*xi* →0) – Σ*pi*log(*pi*/Δ*xi*). (7)

This formulation is very similar to one proposed by Bailey (1990) for measuring *social entropy*, with again the focus on degrees of similarity or equality across social groups or zones.

Among the more direct applications of this for urban systems has been in studying urban sprawl (Cabral et al., 2013). One line has been to measure the degree of fragmentation of ownership. Miceli and Sirmans (2007) argues that this discourages development as real estate developers prefer less dispersed patterns of ownership. Scattered patterns associated with urban sprawl lead to a form of monopoly power that manifests itself through the holdout problem. More broadly urban sprawl is seen as contributing to a variety of social and environmental problems, with higher costs of infrastructure and even greater public health problems (Brueckner, 2000; Nechyba and Walsh, 2004; Frenkel and Ashkenazi, 2009).

While most observers see urban sprawl as posing major problems, it has its defenders. Thus Wassmer (2008) argues that sprawl increases satisfaction with housing and schools, lower crime rates, and greater convenience of car travel, although the latter is a target of those arguing sprawl exacerbates environmental problems. Cabral et al. (2013) see this a matter of tradeoffs. Higher spatial entropy levels are demanding for transport and infrastructure, while lower levels increase levels of inequality and social economic fragmentation.

Unsurprisingly information entropy measures have been used to measure degrees of racial segregation in urban areas for both residences and schools (Mora and Castillo, 2011). While probably the most commonly used measured in these studies is the Theil index shown in Equation (6) above and first proposed for studying school segregation by Theil and Finiizza (1971), with applications such as studying segregation in the San Francisco Bay area (Miller and Quigley, 1990). However, Mora and Castillo argue for the superiority of the de-normalized form of this known as the mutual information index, also due to Theil (1971), which may be more useful for studying decomposability by schools.

Yet further spatial applications include measuring diversity of land use patterns (Walsh, 1977) and spatial settlement distributions (Medvedkov, 1967) as well as spatial patterns of population distribution (Chapman, 1970). Purvis et al. (2019) provide an overview of many of these applications.

1. **Thermodynamic Sustainability**

Most of the models discussed in the previous two sections have relied on the metaphorical information concept of entropy coming from Shannon and Weaver, with the possible exception of Wilson’s development of slow dynamics that draws more directly on Boltzmann. However, another strand of entropic analysis of urban and regional systems relies more on the original ontological approach in which an urban or regional system is seen as being driven by thermodynamics in its original physical sense involving energy transfers and transformations following the Second Law of Thermodynamics. Among those pursuing such an approach have been Rees (1992), Balocco et al. (2004), Zhang et al. (2006), Marchettini et al. (2006), and Purvis et al. (2019).

The focus of most of this research is particularly on the ecological sustainability of urban and regional systems, with viewing them as open dissipative systems experiences inflows and outflows of energy and materials (Georgescu-Roegen, 1971; Prigogine, 1980). While for closed systems entropy increases, with open systems entropy can either increase or decrease if energy and materials flow into the system. This was indeed the Schrödinger (1945) about life, that it was involves an anti-entropic process whereby living things draw in energy and create order and structure as long as they live. A specific term for anti-entropy is *exergy* (Rant, 1956).

Let us then distinguish three concepts: total entropy or *Stotal*, inside entropy or *Si*, and outside entropy or *So*. These are related dynamically according to

d*Stotal*/dt = d*Si*/dt + d*So*/dt, with d*Si*/dt > 0. (8)

However, d*So*/dt can be either positive or negative, so if it is negative and has an absolute value exceeding that of the absolute value exceeding that of *Si*, then total entropy may decline as the system generates order as it draws in energy and materials, only to export them as waste and disorder, with entropy increasing outside the system. As Wackernagel and Rees (1996) put it, “Cities are entropic black holes,” with this raising serious questions about their sustainability as they generate large ecological footprints.

Exergy is often defined as being the maximum amount of useful work possible to reach a maximum entropy state, which means it must be zero if a maximum entropy state is achieved. Rant’s (1956) original formulation was in the context of chemical engineering. If *B* is exergy, *U* is internal energy, *P* is pressure, *V* is volume, *T* is temperature, *S* is entropy, *μi* is the chemical potential of component i, and *Ni* is the moles of component i, then Rant’s formulation is given by

*B* = *U* + *PV* –*TS* – Σ*μiNi*. (9)

This implies, ceteris paribus, that

d*B*/dt ≤ 0 ↔ d*S*/dt ≥ 0, (10)

which highlights the interpretation of exergy as being anti-entropy.[[7]](#footnote-7)

An application of this using a modification of Rant’s equation due to Moran and Sciubba (1994) has been done by Balocco et al. (2004). They study the exergy involved in building construction and real depreciation in the town of Castelnuovo Beardenga near Siena, Italy. This involves also using input-output relations involved with the construction industry. They conclude that more recent buildings are not as efficient as older ones, with those built in 1946-1960 providing the highest sustainability.

Following both Wackernagel and Rees as well as Balocco et al. and also Haken (1988) and Svirzhev (2000), Zhang et al. (2006) engage in an ambitious effort to apply entropy concepts to the study of sustainable development of Ningbo, China, a city of nearly 6 million somewhat south of Shanghai in Zhejiang province. Their effort combines both ontological measures of entropy as well as metaphoric information ones as they break their analysis into four parts. The first two are tied to development and are *sustaining input entropy* and *imposed output energy*, which are basically determined by production. The second two are considered to be part of the metabolism of the urban system, *regenerative metabolism* and *destructive metabolism,* which are tied to the generation of pollution and its cleanup. This becomes a measure of harmony with the environment. The outcome of the first gives the developmental degree while the second gives the harmony degree. They estimate these for the 1996-2003 period and find that these two measures were generally going in opposite directions, with the developmental degree rising (associated with declining entropy) as the harmony degree was declining (associated with rising entropy). This poses the problem of sustainability of urban development in China quite sharply.

Marchettini et al. (2006)[[8]](#footnote-8) consider this approach from a more general level, drawing on ideas due to Morin (1995) regarding autonomy versus dependence of systems on their environment, while using the dissipative structures approach of open systems associated with Prigogine (1980). They see urban systems evolving between extremes of autarchy and globalization. However, they argue that in the end neither of these extremes is sustainable, In their advocacy of a balanced path they emphasize how urban and regional systems are ecosystems that operate on the basis of energy flows (Odum, 1969) within a set of complex wholes emerging from a set of interacting micro-level components (Ulanowicz, 1986).

1. **The Anti-Entropic Alternative**

Pushing against this entropic version of the structure of urban and regional systems is a power law version of such structuring, at least for certain cases and situations. Arguably this is dealt with in the entropy framework, given the matter of the balance between exergyy and entropy in urban and regional systems. Most of the systems and measures up until now have involved essentially internal relations or distributions within urban or regional systems. But when one considers higher level distributional systems the entropy relation may break down or even become completely irrelevant.

One way that anti-entropic forces can manifest themselves is by the appearance of power law distributions (Rosser, 2016),[[9]](#footnote-9) with substantial evidence that city sizes may follow such distributions (Gabaix, 1999). Pareto (1897) identified the concept of power law distributions. For *P* is population, *r* is rank, and A and α are constants, then

*rPr*α = A, (11)

which can be put into log-log form, which is linear,

ln *r* = ln A – α(ln *Pr*). (12)

We note that for the special case of α = 1, the population of entity of rank *r* becomes

*Pr* = *P1*/*r*, (13)

Which was labeled the *rank-size rule* by Auerbacch (1913) and would later come to be known as Zipf’s Law, argued to hold for many distributions (Zipf, 1949).[[10]](#footnote-10)

The issue of whether or not city size distributions follow Zipf’s Law and thus obey the rank-size rule has been a matter of ongoing debate since Auerbach (1913) first proposed it and Lotka (1925) questioned it. Some, especially urban geographers (Berry and Okulicz-Kozryn, 2012) have argued that it is a universal law. Others, more often economists, have questioned it, arguing that there is no clear reason why it should be followed, even if there city sizes may well exhibit power law distributions (Batten, 2001; Fujita et al., 1999), although Gabaix (1999) argues that Zipf’s Law arises in the limit if Gibrat’s Law holds that growth rates are independent of city sizes.

Batten (2001) in particular shows US city size distributions exhibiting power law distributions from 1790 to the present, even if not exactly the rank-size rule (with the fact that Los Angeles is substantially larger than half the size of New York an example why it might not hold). Nitsch (2005) carried out a meta-study of past empirical studies, observing a wide range of findings across studies, but when looking at them in the aggregate they found a mean of α = 1.08, quite close to the Zipf value. Berry and Okulicz-Kozaryn (2012) argue that the variations in estimates are due to not using consistent measures of urban regions across studies, and if the largest such measures are used of megalopolises, then Zipf’s Law and the rank-size rule holds fully. In any case, whether it does or not, the evidence is strong that city size distributions are power law distributed, showing a domination by anti-entropic forces for this part of urban and regional systems.

A possible foundation for these anti-entropic processes that can generate power law distributional outcomes is economies of scale, long known to be a foundation also of economic complexity (Arthur, 1994). Urban systems in particular can exhibit as many as three different kinds of economies of scale: internal firm level economies (Marshall, 1879), localization economies involving external agglomeration between firms in a single industry (Marshall, 1919), and urbanization economies that involve external agglomeration economies spilling across industries (Hoover and Vernon, 1959).

Rigorous models of how increases in agglomerative tendencies can overcome congestion effects can destabilize an equilibrium of equal population distribution, essentially a maximum entropy outcome, and lead to the rise of urban concentrations are due to Papageorgiu and Smith (1983) and Weidlich and Haag (1987). However these models have since been superseded by “new economic geography” ones that emphasize economies of scale arising in within monopolistic competition as analyzed by Dixit and Stiglitz (1977). While Fujita (1988) initiated using this for modeling urban and regional systems, Krugman’s (1991) approach received the most attention and influence (Rosser, 2011).

1. **Entropy, Complexity, and Self-Organization of Urban and Regional Systems**

This brings us to a realization that the interaction between entropic and anti-entropic forces within urban and regional systems can generate complexity that underlies emergence of higher ordered structural patterns through self-organization as bifurcation points are encountered within nonlinear dynamics of the systems that lead to morphogenetic structural transformations (Rosser, 1990, 1991; Krugman, 1996; Portugali, 1999). This can be seen by looking at how these systems operate from the perspective dynamic complexity, which Day (1994) defined as systems endogenously not converging on a steady state or exponential growth. Such complexity is known to take four forms: cybernetics, catastrophe theory, chaos theory, and agent-based complexity (Rosser, 1999). All these forms can be seen to have operated within urban and regional systems.

The most important model of urban dynamics based on a cybernetics was due to Jay Forrester (1969) in his *Urban Dynamics*, although he labeled his approach to be part of *systems dynamics* theory. This involved a set of nonlinear difference equations with complicated interconnections with each other involving positive and negative feedback effects. When simulated it exhibited structural breaks and sudden changes at certain points, with the system too complicated for discovering these by analysis, rather requiring simulation instead.

Much more widespread have been studies of structural changes in urban and regional and more general spatial systems using catastrophe theory. Amson (1974) initiated the use of catastrophe theory in such systems, examining rent and “opulence” (attractiveness) determinants of urban density using a cusp catastrophe model. Mees (1975) modeled the revival of cities in medieval Europe as a butterfly catastrophe. Wilson (1976) modeled transportation modal choice as a fold catastrophe, and drawing on the entropic retail model, Poston and Wilson (1977) did so for retail center size.[[11]](#footnote-11) Isard (1977) initiated the study of agglomeration effects bringing about the sudden emergence of cities in models balancing urban and rural areas using the cusp catastrophe, with Casetti (1980) and Dendrinos (1980) following. Dendrinos (1978, 1979) used higher order catastrophe models to study industrial-residential dynamics and slum formation in cities. Puu (1979, 1981) did so as well to study structural changes in regional trading patterns. Nijkamp and Reggiani (1988) showed how an optimal control model of nonlinear dynamic spatial interaction can generate a catastrophe theoretic interpretation.

The application of chaos theory to the study of complex urban and regional dynamics was initiated by Beaumont et al. (1981) for intraurban residential and retail dynamics, again drawing on the entropic intraurban model. White (1985) combined this model with ideas from synergetics (Haken, 1977) to [[12]](#footnote-12)show self-organization arising from chaotic fluctuations near bifurcations points. A series of papers and books emphasized interregional migration or more general population dynamics (Rogerson, 1986; Day et al., 1987; Dendrinos, 1982; Dendrinos and Sonis, 1990). Another area of study was chaotic dynamics in interregional business cycle models (Puu, 1989, 1990). There have also been studies of chaotic dynamics in extended versions of the new economic geography core-periphery models based on monopolistic competition (Currie and Kubin, 2006; Commendatore et al., 2007).

Finally, it turns out that the very initiation of agent-based complexity models came out of efforts to model the emergence of racial segregation in cities by Schelling (1971, 1978). These changes can be measured by entropic methods. Curiously, Schelling did not use either analytic models or computer simulation, but instead played a game on a 19 by 19 Go board with black and white stones, simply assuming small local differences in desires to live next to people like one or not. A high entropy beginning of integration ends up with emergence of a low entropy segregated pattern. Schelling’s model has been studied since in many variations and contexts and found to be highly robust. Zhang (2004) considered it as an evolutionary game on a lattice torus, while Fagiolo et al. (2007) as a network model. Such models have a similarity to the cybernetics models, except that they more clearly rely on generating higher-order self-organizaton emerging from low level agents interacting with each other according to strictly local effects, a foundational complexity approach.

**VIII. Conclusions**

It is completely natural that both entropy and complexity are deeply involved in the dynamics and spatial structures of urban and regional systems. The spatial nature of such systems opens them to having local neighborhood effects being very important, which is foundational for advanced views of complexity and the ubiquity of external agglomeration effects underlie nonlinearities that furthermore lead to dynamic complexities of various sorts, including catastrophic discontinuities and chaotic dynamics.

As open systems, complexity is further enhanced by the dissipative nature of urban and regional systems. They are subject to the competition between entropic and anti-entropic forces that interact to stimulate complex dynamics. This is especially the case for the ontological thermodynamics of urban and regional systems operating as ecosystems.

However, metaphorical entropy measures based on Shannon information entropy have proven useful in understanding and modeling a variety of aspects of urban and regional systems. This includes both spatial patterns as well as sociological structures such as racial segregation, which have also been found to exhibit complex dynamics as with the Schelling model. Few areas of economics or the broader social sciences exhibit so many instances of complex dynamics enhanced by entropic forces as urban and regional systems.

This interaction calls for a new world view. As Jeremy Rifkin (1981, p. 256) puts it, “In the end, our individual present rests forever in the collective soul of the unfolding process itself. To conserve as best we can the fixed endowment that was left to us, and to respect as best we can the natural rhythm that governs the becoming process, is to express our ultimate love for all life that preceded us and all life that will follow.”

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1. Purvis et al. (2019) argue for a third type of entropy, “figurative,” which suggests an increasing disorder or randomness. However, here this form will be considered to be subsumed into the other two, especially in the first Samuelson (1972) provides a critique of some uses of entropy in economic models as well as Kovalev (2016). Foley (1994) provides an entropy-based alternative to economic general equilibrium, and Stutzer (2000) provides an entropy-based derivation of the Black-Scholes formula for options pricing in finance. [↑](#footnote-ref-1)
2. Formal modeling of agglomeration in urban systems is due to Fujita (1988) and Krugman ((991) based on Dixit and Stiglitz (1977). See also Fujita et al. (1999). [↑](#footnote-ref-2)
3. Carnot’s book was long hard to find a copy of and long had little direct influence on the development of steam engines, although eventually its implication that having a greater difference in temperature between the source and the sink could increase the efficiency of such engines was acted on by people such as Joseph Diesel in developing improved steam engines (Georgescu-Roegen, 1971). [↑](#footnote-ref-3)
4. Gibbs (1902) proposed this to be a strictly probabilistic formulation that is a purely mathematical statement, leading to his famous remark that “mathematics is a language.” [↑](#footnote-ref-4)
5. The distinction between ontological and metaphorical entropy is due to Rosser (2016). Lotka (1922) argued that evolution is fundamentally driven by an ontological thermodynamic process based on the law of entropy. Brooks et al., (1989) see metaphorical information entropy as useful for understanding biological evolution. [↑](#footnote-ref-5)
6. For alternative systems providing similar possibilities see Allen and Sanglier (1979) and Nijkamp and Reggiani (1988), with Rosser (2011) providing a broad overview. [↑](#footnote-ref-6)
7. It is also sometimes known as *negentropy*, for “negative entropy.” [↑](#footnote-ref-7)
8. Ironically Marchettini and coauthors are in the same institute at the University of Siena as the coauthors of Balocco, but neither group cites the work of the other. [↑](#footnote-ref-8)
9. Yakovenko and Rossser (2009) show that the US income distribution can be characterized by a combination of entropic and anti-entropic mechanisms at different levels, with entropic dominating for the lower 98 percent or so of the distribution, but that more highly skewed anti-entropic power laws dominate for the top 2 percent of the distribution. [↑](#footnote-ref-9)
10. Gabaix and Ioannides (2004) argue that Simon (1955) first provided a formal way to estimate power law distributions for urban sizes. [↑](#footnote-ref-10)
11. Wilson (1981) provides an early overview of many of these models. Dendrinos and Rosser (1992) show how many are linked. [↑](#footnote-ref-11)
12. Extensions of this to fractal synergetic models of self-organization of urban hierarchies are due to Fotheringham et al, (1989) and Rosser (1994). [↑](#footnote-ref-12)