ENTROPY AND ECONOPHYSICS

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Abstract: Entropy is a central concept of statistical mechanics, which is the main branch of physics that underlies econophysics, the application of physics concepts to understand economic phenomena. It enters into econophysics both in an ontological way through the Second Law of Thermodynamics as this drives the world economy from its ecological foundations as solar energy passes through food chains in dissipative process of entropy rising and production fundamentally involving the replacement of lower entropy energy states with higher entropy ones. In contrast the mathematics of entropy as appearing in information theory becomes the basis for modeling financial market dynamics as well as income and wealth distribution dynamics. It also provides the basis for an alternative view of stochastic price equilibria in economics, as well providing a crucial link between econophysics and sociophysics, keeping in mind the essential unity of the various concepts of entropy.

"I have come over the years to have some impatience and boredom with those who try to find an analogue of the entropy of Clausius or Boltzman or Shannon to put into economic theory. It is the *mathematical* structure of *classical* (phenomenological, macroscopic, nonstochastic) *thermodynamics* that has isomorphisms with *theoretical economics* ."

Paul A. Samuelson, 1990, "Gibbs in Economics," p. 263 [1]

"…throughout his [Samuelson's] career…the master of scientific rhetoric, continuously hinting at parallels between neoclassical theory and twentieth century physics, and just as consciously denying them, usually in the same article."

Philip Mirowski, 1989, "How not to do things with metaphors: Paul Samuelson and the science of neoclassical economics," p. 186. [2]

1.0 Introduction

The problematic role of entropy in econophysics is highlighted by the quotations presented above: that the most influential economist of the twentieth century, Paul A. Samuelson, played it both ways regarding the role of the concept of entropy in the development of economic theory, and more broadly the role of physics in economics. While he regularly ridiculed applications of the entropy concept in economics, he more powerfully than any other economist imposed concepts drawn on physics onto standard neoclassical economics, including that important part of econophysics, statistical mechanics, a contradiction pointed out so forcefully by Mirowski.

The term "econophysics" was introduced verbally by H. Eugene Stanley at a conference in Kolkata in 1995, and in print by Mantegna and Stanley (2000) [3] who identified it with physicists applying ideas from physics into economics. This formulation becomes problematic when we understand that people educated as physicists have long been doing this, with Samuelson himself a leading example, along with the one of those who received a Nobel Prize in economics before him, Jan Tinbergen [4], whose major professor was Paul Ehrenfest, who formulated the "ergodic hypothesis" with his wife in 1911 [5], drawing on the work of his major professor, Ludwig Boltzmann [6].[1] Boltzmann linked the study of statistical mechanics to the concept of entropy as developed initially by Clausius [8], who in turn was inspired by the work on the thermodynamics of steam engines by Carnot [9] in 1824. The simplest formulation of the Law of Entropy took the form of the Second Law of Thermodynamics: that in a closed thermodynamical system entropy increases.

Given that these borrowings from physics into economics have long predated the more recent movement of physicists to apply their models to economics, we shall expand the concept of economics irrespective of whether these applications were done by people who were primarily physicists, primarily economists, or who were arguably both, with many important economists having originally trained as physicists, with Tinbergen as the student of Ehrenfest being an example.

Regarding the specific application of the entropy idea into economics and hence as a form of econophysics, we shall distinguish between two basic approaches. One may be labeled ontological while the other can be viewed as metaphorical, although some involved in this have at times confused these two such as the energeticists, Helm, Winiarski, and Ostwald, as described by Mirowski [10].In the ontological formulation, the foundation of the economy is seen being physical and biological processes driven by energy, with the Second Law of Thermodynamics serving as a key organizing principle for this foundation, with Georgescu-Roegen [11] the most influential exponent of this idea, even as he was also a critic of it, as noted by Rosser[12]. This view follows more the tradition of Carnot and Clausius, but also depends on the work of Boltzmann as modified by Gibbs [13] in statistical mechanics, with the

[1] See Rosser [7] for further discussion of the development of the ergodic hypothesis and its relationship to economics.

Boltzmann-Gibbs formulation of the Law of Entropy. This approach has its greatest advocates among ecological economists, some of whom speak of this view as representing “biophysical economics” [14].

The metaphorical approach draws more on the information formulation of entropy due to Claude Shannon [15], with applications across finance and equilibrium theory, many of these more closely tied to modern econophysics. Ultimately these two concepts of entropy share common mathematics of probability distributions of logarithms of products of possible states of the world, even as they have considerably different applications. While much of modern econophysics is more concerned with other matters such as power law distributions of variables, the entropy concept enters into many applications of econophysics, with important new approaches to economics relying on these more metaphorical formulations.[2]

1.1 Unity of the Core Entropy Concepts

The most widely used form of the Boltzmann equation for entropy is on his grave, although he never wrote it down in that way [19]. It involves *W,* the thermodynamic probability of an aggregate state of a system of gas molecules, with *k* the Boltzmann constant, and *S* being entropy. It takes the form

$$S = k \ln W. \tag{1}$$

Given *N* microscopic states of the system, the probability of a gas molecule being in the ith state is N_i/N. *W* is then given by [20]

[2] We note that there are now a variety of extensions of the more basic Boltzmann-Gibbs and Shannon versions of entropy, including Rényi [16], and Tsallis [17] (this latter more closely tied to the study of power law distributions), with various efforts at generalizing these being made such as by Thurner and Hanel [18]. However, we shall not focus on these and note that most of these reduce to the simpler forms asymptotically as certain modifying parameters approach infinity, even as we recognize that they may well be useful for future applications.

$$W = N!/\Pi N_i!. \tag{2}$$

This means that Boltzmann entropy can be rewritten as

$$S = k \ln (N!/\Pi N_i!). \tag{3}$$

Basic Shannon entropy is given by H of the probability distribution of states of informational uncertainty for message i. of $H(p_1 \ldots p_n)$. This then equals [14]

$$H(p_1 \ldots p_n) = -k \Sigma p_i \ln p_i. \tag{4}$$

Recognizing that $p_i = N_i/N$, the basic unity of these two concepts appear as N increases, which leads the Boltzmann formula in (3) to approach [11]

$$S = -kN\Sigma p_i \ln p_i, \tag{5}$$

which means that in the limit as N approaches infinity, Boltzmann entropy is proportional to Shannon entropy.

2.0 Ontological Entropy and Econophysics

2.1 Entropy as the Fundamental Limit to Growth

The ontological approach to econophysics derives from the direct and foundational role of energy in the economy, not merely for industrial production or providing for electricity or transportation, but at the ecological or biophysical level, that of solar energy driving the global biosphere. This is more a return to the Carnot and Clausius view of thermodynamics, where the continued incoming of solar energy shows the openness of the earth's system that allows it to avoid the law of entropy as long as the sun lasts [11] [21].[3] However, that arriving solar energy itself is finite and thus provides a direct limit on economic activity that depends on the ecosystems through which the solar energy dissipates in the food chains that are driven by that energy. In addition, Georgescu-Roegen [11] extended this argument to broader material resource inputs, arguing that they are also subject to a form of the law of entropy as well that provides further limits on the economy. More broadly for him [11, p. 281] "the economic process consists of a continuous transformation of low entropy into high entropy, that is, into *irrevocable waste*, or, with a topical term, into pollution."

While variations of this argument have become highly influential, especially in ecological economics as with Martinez-Alier [24], it has faced sharp criticisms as well. Thus,Gerelli [25] argues that the scale of the solar input is such that it is orders of magnitude beyond really limiting the world economy, with many other more mundane constraints more relevant in the short run. Nordhaus [26] estimated entropy to be as many as 12 orders of magnitude below technology as a limit to growth, with Young [27] weighing in similarly. In that regard the drawdown of stored energy sources and their limits such as with fossil fuels may be more relevant with the pollution from using them even more limiting as with such outcomes as climate change arising from the burning of such fuels releasing their stored carbon dioxide. Other critics have emphasized either the limitless ingenuity of the human mind such as Julian Simon, who argued that [28, p. 347] "those who view the relevant universe as unbounded view the second law of thermodynamics as irrelevant to the discussion."

Another important figure in this line of argument was Alfred J. Lotka [29], the father of the concept of predator-prey cycles. Lotka argued that the law of entropy is a deep driving force in evolution, a source of a teleological directedness of the process towards greater complexity. He saw

[3] Georgescu-Roegen [11] in particular strongly relied on the argument of Schrodinger, Chapter 6 [22] regarding how life is ultimately an anti-entropic process based on organisms being open systems able to draw in both matter and energy while they live, with in this sense the death of organisms representing the ultimate victory of entropy. An alternative is to more directly follow Carnot and Clausius in emphasizing the role of the steam engine in the modern economy as in Cockshott et al [23].

this as the fundamental physical foundation of biology that needed to be studied mathematically, and he in turn saw the economy as deriving from the ecosystem as the more recent ecological economist have. Ironically Lotka was a tremendous influence on Paul Samuelson, who cited him prominently in his magnum opus, *Foundations of Economic Analysis* [30], although more for his categorization of the stability conditions of linear systems rather than for his arguments regarding the law of entropy or its relation to the economy.

2.2 Entropy and the Energy View of Economic Value

Closely related to arguing that energy flows dissipating as the law of entropy operates are the foundation of the economy is the idea that either energy or some measure of entropy should be the basis for measuring value in an economy. This was first propsed by “energeticist” physicists of the late 19^{th} and early 20^{th} centuries. Thus Helm [31] and Winiarski [32] argued that gold was “socio-biological energy.” Closer to the entropy argument was Ostwald [33] whose view was that conversion factors based on the physical availability of specific forms of energy was the key to fundamental value determination. Extending this, Julius Davidson [34] argued that the law of diminishing returns in economics[4] was ultimately based on the law of entropy. Much later Davis [35] would argue that the utility of money was a form of “economic entropy,” although Lisman [36] noted that this was not operationally equivalent to how the law of thermodynamics works in physics, and Samuelson [37] simply dismissed these arguments as being “crackpot.”

[4] The law of diminishing (marginal) returns or productivity is probably the only so-called “law” in economics for which no counterexample has been found.

Interestingly some of those who supported the idea of entropy playing a fundamental ontological role in economics also had issues with such approaches to value. Lotka [29, p.355] noted that,

> "The physical process is a typical case of 'trigger action' in which the ratio of energy set free to energy applied is subject to no restricting general law whatsoever (e.g. a touch of the finger upon a switch may set off tons of dynamite). In contrast with the case of thermodynamics conversion factors, the proportionality factor is here determined by the particular mechanism employed."

Likewise for Georgescu-Roegen [11], while he saw entropy as the ultimate limit to growth, he did not see it as all that useful for determining value, which he saw as ultimately coming from utility. Thus, nobody wants the low entropy poisonous mushroom and some people value more highly the high entropy beaten egg to the low entropy raw egg. These are matters of utility, and while Georgescu-Roegen did not see utility (or marginal utility to be more precise) as the sole source of value as did the subjectivist theorists of the Austrian School, he certainly saw it as very important and was a major developer of modern utility theory early in his career.[5]

3.0 Metaphorical Entropy and Econophysics

3.1 Financial Modeling

From the beginning of the coining of the term *econophysics* [3], a major focus has been on applying physics concepts to financial market dynamics.[6] Central to all financial analysis is concern with how to model price, risk, and uncertainty. This inevitably involves study of probability distributions and stochastic processes. Unsurprisingly statistical physics has provided models and inspiration for doing

[5] Rosser [38] provides further discussion of this debate.
[6] Good overviews can be found in [39], [40], and [41].

this, including at times the concept of entropy from Boltzmann and Gibbs are drawn on. However, we thermodynamical processes driving the economy whether through industrial production or through foundational biophysical systems it is the mathematics of Shannon and other entropies that can be used to understand these stochastic processes in a metaphorical fashion.

In this regard the analogy to Shannon entropy, which some of these models draw on specifically, it can be argued that Shannon entropy itself is a metaphor in a way that Boltzmann-Gibbs entropy is not, or especially that Carnot-Clausius entropy Is not. Again, the latter is an ontological foundation of physical phenomena, the dissipation of heat energy in mechanical processes initially, but then, inspired by Maxwell, providing a mathematics to explain heat itself Shannon entropy deals with something more abstract, information, although that certainly has its real world uses, as does financial modeling. However, it is harder to say that the law of entropy itself is what is driving these phenomena rather than that the mathematics of entropy is useful for understanding or explaining them.

On the title page of his *Foundations of Economic Analysis* [30], Paul Samuelson famously quoted Gibbs as saying, “Mathematics is a language.” That it certainly is. But in the case of Shannon entropy as well as financial models based on entropy mathematics, it is a metaphor rather than a linguistic ontology.

Econophysics has been dragged into the old argument between quantifiable risk and non-quantifiable uncertainty that was initialliy posed in 1921 independently by Keynes [42] and Knight [43]. Drawing on much discussion from various econophysicists, Schinkus [44] argues that econophysicists are more inclined than regular economists to approach data without preconceptions regarding distributions or parameter values, although it may be that they may be more inclined to draw on ideas from physics, with entropy among those in connection with financial modeling. Thus, Dionisio et al [45, p. 161] argue that:

"Entropy is a measure of dispersion, uncertainty, disorder and diversification used in dynamic process, in statistics and information theory, and has been increasingly adopted in financial theory."

In this case, entropy is associated with the more traditional forms of financial modeling than with what has probably been the most central theme of financial econophysics. This latter has been applying power law distributions especially to financial market distributions, this effort initiated by Mandelbrot [46], with many econophysicists criticizing economists for their propensity to rely on Gaussian distributions to explain financial markets, with this tradition going back to Bachelier [47]. As it is applications of the law of entropy using Shannon entropy or Boltzmann-Gibbs distributions easily fit into explaining or modeling distributions that rely on lognormality, which are easily consistent with Gaussian approaches. While we know that ultimately these entropies are essentially identical mathematically, the real difference is that one we believe is driven to maximization as a law of physics whereas in the more metaphorical ones observing an exremum for entropy is simply a useful mathematical condition.

Someone drawing on both of the main measures of entropy in order to develop core financial theory in the form of the Black-Scholes options pricing formula [48] is Michael J. Stutzer [49],[50]. In the second of these he used Shannon entropy for his generalization of the link, after pointing out that Cozzolino and Zahner [51] in 1973 had used Shannon entropy to derive lognormal stock price distributions, the same year that Black and Scholes [48] published their result without directly relying on any entropy mathematics. For his generalization Stutzer [50] posed the problem in discrete form as considering a stock market price process given by

$$\Delta p/p = \mu\Delta t + \sigma\surd\Delta t\Delta z, \tag{6}$$

where *p* is price, *t* is the time interval, and the second term on the right hand side is the random shock, with these distributed ~ $N(0, \Delta t)$. With *Q* as quantity, $r\Delta t$ the riskless rate of return, and P the actual conditional risk density distribution, a central focus is the conditional risk neutral given by *dQ/dP*.

From these one considers the relative entropy minimizing conditional risk neutral density that in effect maximizes order

$$arg\ min_{dQ/dP} \int log\ dQ/dP\ dQ, \tag{7}$$

subject to a martingale restriction given by

$$r\Delta t - E[(\Delta p/p)(dQ/dP] = 0, \tag{8}$$

From this he shows that when asset returns are IID with normally distributed shocks as given above, the martingale product density formed from the relative entropy minimizing conditional risk is that used to calculate the Black-Scholes option pricing formula. He recognizes that this does not easily generalize to non-Gaussian distributions such as the power law ones much studied by econophysicists, suggesting a weaker approach using Generalized Auto Regressive Conditional Heteroskedastic (GARCH) processes.

We note further that not all econophysics models of financial markets have relied on ideas coming from statistical mechanics, whether involving entropy or not. The most prominent alternative source for such modeling has derived from geophysics, with Sornette [52] being a major advocate and developer of this approach. Some of this work has proven to be controversial with substantial debates arising around it [54], but these matters are beyond this paper, as are some broader debates regarding econophysics that have nothing to do with the issue of the use of the idea of entropy in such models [53], [54], [55].

3.2 Modeling Wealth and Income Distribution Dynamics

Another area that has drawn the attention of econophysicists with the concept of entropy at least partly involved has been that of studying wealth and income distribution. Interestingly by perhaps more than in other areas we find that the relationship between entropy-based non-power law distributions and power law distributions plays a central role in the modeling of these dynamical systems. In particular it increasingly looks as if while wealth largely reflects power law distributions, income distribution may be a combination, with entropy-related Boltzmann-Gibbs distributions best explaining income distribution for the poorest 97-98 percent, whereas a Pareto power law distribution may do better for the top level of income, where wealth dynamics may play a more important role [56], [57].

Awareness of the possibility of using entropy ideas in the measurement of income distribution began with economists looking for generalizations of the various competing measures that have been used for studying income distributions. Thus in 1981, Cowell and Kuga [58] sought a generalized axiomatic formulation for additive measures of income distribution. They found that by adding two axioms to the usual approach they were able to show that a generalized entropy approach could subsume the widely studied Atkinson measure [59] and Theil measure [60]. While the Atkinson measure has been more widely used and is able to distinguish skewness of tails, the Theil may have more generality. Bourgignon [60] shows that it is the only decomposable "income-weighted" inequality measure that is zero homogeneous. Cowell and Kuga [58] show that adding a sensitivity axiom to their others yields the Theil index as the only one that is derivable from a generalized entropy concept.

These early discussions also involved strong claims regarding the difficulties of linking entropy measures with power law distributions, claims that now look to be overdone to some extent. Thus we find Montroll and Schlesinger [61, p. 209] claiming that:

> "The derivation of distributions with inverse power tails from maximum entropy formalism would be a consequence only of an unconventional auxiliary condition that involves the specification of the average of a complicated logarithmic function."

As we shall see, this statement seems to be somewhat overdone, although indeed logarithmic functions are involved in the relationship between the two, which is not surprising given that entropy measures are essentially logarithmic.

The power law distribution approach dominates discussion of wealth distribution dynamics, as it does financial market dynamics. The father of this approach was Vilfredo Pareto [62], who was initially trained as an engineer, but then became a socio-economist as his theory involved the relationship between social classes over time. Very appropriately Pareto's original motivation and focus of study was in fact income distribution. Like some later econophysicists he claimed a universal truth associated with an estimated income distribution parameter. He was wrong, especially given that his theory fits better wealth distributions rather than income distributions, where, as noted above. Pareto argued incorrectly that his supposedly universal coefficient for the power law explanation of income distribution, which fit into his theory of the "circulation of the elites," in which nothing could be done to equalize income because the political process would simply involve substituting one power elite for another with no noticeable change in the income distribution. But we must recognize that he formulated this view at the end of the 19th century, when there had been a century of no major changes in the socio-economic structure anywhere. Needless to say, not too long afterwards there were large changes in the distribution, even as his method went "underground," only to be revived for other uses such as describing urban metropolitan size distributions [63].

The modern concern with income distribution based on power law physics concepts from Pareto was due to a sociologist, John Angle [64]. After the appearance of current econophysics, many stepped forward to apply power law distributions to study the dynamics of wealth distributions. Drawing on the work of Pareto, who mistakenly thought he had found a universal coefficient for income distribution, econophysicists found that current wealth distributions fit Pareto's power law view [65]. [66].

At this point the question needs to be considered as to whether we are dealing with ontological as opposed to "merely" metaphorical models in these matters. We know that there are stochastic tendencies for wealth and income dynamics, but it is not at all obvious that the various apparent imperatives for entropy maximization or minimization are actually driving outcomes. Nevertheless many studying these matters see thermodynamical processes underlying basic tendencies of wealth and income distribution dynamics. Such processes are not quite as direct as the ontological direction based on Carnot's steam engines, but derive from broader tendencies of wealth and income distribution dynamics occurring in the absence of substantial changes in public policy regarding distributional policies.

Pareto was mistaken in his original proposal. He thought that he had found a universal law of income distribution that fit with his theory of the "circulation of the elites," within which it did not matter which elite group was ruling society, the underlying distribution of income would not change. He was wrong. The legacy of his approach has been in the study of wealth distributions, where his presentation of power laws is now understood to explain wealth distributions rather than income distributions.

The Pareto distribution is given by:

$$N = A/x^{\alpha}, \tag{9}$$

where *N* is the number of observations above *x*, and *A* and *α* are constants. This includes as special cases a wide variety of other forms that underly many econophysics models. The special case when $\alpha = 1$ leads to "Zipf's Law," [67], widely viewed to describe urban size distributions as well as many others, although how far this "law" applies is a matter of ongoing debate.

Yakovenko and Rosser [57] present a unified income distribution analysis combining an entropic Boltzmann-Gibbs formulation for lower income distribution with a Paretian power law distribution for the highest levels of income. The model makes a heroic assumption of conservation of money or income or wealth, which empirically is not unreasonable for the United States since the mid-1970s for median levels, even as the top strata have seen growing levels. But this fits with the combination of a lognormal entropic model for the majority of the population with regard to income, even as the top level of the income distribution seems to follow a wealth dynamic following a Paretian power law distribution.

Assuming a conservation of money, *m*, the entropically based Boltzmann-Gibbs equilibrium distribution is given by the probability, *P*, that the level will be *m*, given by:

$$P(\mathrm{m}) = \mathrm{c}e^{-\mathrm{m}/T_m}, \tag{10}$$

where c is a normalizing constant, and T_m is the "money temperature" in thermodynamic terms, which is equal to the money supply per capita. This describes the lower portion of the income distribution.

Assuming a fixed rate of proportional money transfers with this equal to *γ,* the stationary distribution of money (income) is related to the Gamma distribution form that differs from the Boltzmann-Gibbs by having a power-law prefactor, m^{β},where

$$\beta = -1 - \ln 2/\ln(1 - \gamma). \tag{11}$$

This relates the Boltzmann-Gibbs form to a power law equivalent more simply than supposed by Montrell and Schlesinger [61]. This formulation that shows the connection between the two conceptualizations of wealth and income distributions is given by:

$$P(m) = cm^{\beta}e^{-m/T}. \tag{12}$$

This represents the stationary distribution, but allowing *m* to grow stochastically disconnects the outcome from the maximum entropy solution [68]. The stationary distribution under these conditions becomes a mean-field case governed by a Fokker-Planck equation, which is neither Boltzmann-Gibbs nor Gamma, but is a version of a generalized Lotka-Volterra distribution, with *w* the wealth per person, *J* is the average transfer between agents, and σ is the standard deviation, and is given by [69]:

$$P(w) = c[(e^{-J/\sigma\sigma w})/(w^{2+J/\sigma\sigma})]. \tag{13}$$

So it is possible to combine an entropic Boltzmann-Gibbs formulation for the lower part of the income distribution with a power law form for its upper end, which corresponds to the wealth dynamics formulation deriving ironically from Pareto, given that he originally thought his conceptualization was a universal law of income distribution. His formulation would be countered soon after by Bachelier [47], but we now see the two conjoined to provide an empirical explanation of income distribution that has deep roots in Marxist and other classical economic formulations regarding socio-economic class dynamics [23].

3.3 Entropy and General Equilibrium Value

Moving to the heart of economics entropy has been proposed as an alternative to the conventional Arrow-Debreu explanation of value. That standard view has equilibrium being a vector of prices that are fixed points. The entropic alternative recognizes the reality of a stochastic world in which equilibrium is better depicted as a probability distribution of prices as prices are never the same everywhere at any point in time for any commodity except as measure zero accident. An early expression of this idea is due to Hans Föllmer [70]. A fuller development of this has been due to Foley [71], later extended by Foley and Smith [72].

The basic Foley [71] model involves strong assumptions such as that all possible transactions within an economy have equal probability. However his solution involves a statistical distribution of behaviors in the economy where a particular transaction is inversely proportional to the exponential of its equilibrium entropy price, with this coming from a maximum Botlzmann-Gibbs entropy set of shadow prices. Walrasian general equilibrium is a special case of this model when "temperature" is zero. The more general form lacks the usual welfare implications, and it allows for the possibility of negative prices as in the case of Herodotus auctions [73].[7]

Let there be *m* commodities, *n* agents of type *k* who achieve a transaction *x* of which there are is $h^k[x]$ proportion of agents type *k* out of *r* who do transaction *x* out of an offer set *A*, of which there are *mn*. *Multiplicity* of an assignment for *n* agents assigned to *S* actions, each of them *s*, is given by:

$$W[n_{s]}] = n!/(n_1!\ldots n_s!\ldots n_S!). \quad (14)$$

Shannon entropy of this multiplicity is given by:

$$H\{h^k[x]\} = -\Sigma_{k=1}{}^r W^k \Sigma_{x\varepsilon Ak} h^k[x]x = 0. \quad (15)$$

[7] Herodotus described a marriage auction in Babylon with descending prices for potential brides. The most desirable would go for positive prices, but the auction allowed for negative prices for the least desirable potential brides. This contrasts with most societies where there is either a positive bride price or a positive groom price, more often described as a "dowry." The problem of negative prices is often obfuscated by declaring two separate markets, such as one to supply water when it is scarce and a different one to remove it when it is flooding. But the Babylonian bride market described by Herodotus makes it clear that there can be unified markets with both positive and negative prices.

Maximizing this entropic formulation subject to the appropriate feasibility constraints, which if non-empty, gives the unique canonical Gibbs solution:

$$H^k[x] = exp[-\Pi x]/\Sigma_x exp[-\Pi x], \qquad (16)$$

where Π are vectors of the entropy shadow prices.

3.4 Entropy Between Econophysics and Sociophysics

Another metaphorical use of entropy concepts has been in conjunction with that close relative of econophysics, *sociophysics*. Initially coined by Galam et al [74], its roots predated this neologism in the form of *sociodynamics* as developed by Weidlich and Haag [75]. A major emphasis of this sociophysics is on modeling group dynamics including herding. A solution favored by Weidlich and Haag is the master equation, used especially for studying migration patterns, among other phenomena. When constraints do not uniquely solve the stochastic model of this equation, an nth order Markov process can emerge as the unique maximum entropy solution [76].

While not as developed as econophysics, sociophysics has followed its founding by Galam along with Weidlich and Haag along a variety of paths, with Chakrabarti et al [77] providing a fine overview of these investigations. Both the possibilities of applying the entropy concept to this approach have been studied in depth by Mimkes [78] in [77], who also strives to extend his analysis to all of the social sciences. In his formulation we see a return to the question of ontological versus metaphorical applications of the entropy concept as Mimkes ties entropy to the fundamental nature of the production function. While this conjures up the vision of Georgescu-Roegen [11] where the actual processes of the economy are fundamentally a working out of the Second Law of Thermodynamics, Mimkes eventually retreats to a more metaphorical application where it is the mathematical formulation of entropy as a descriptive device for data on distributional outcomes in the economy that is the prime focus of the analysis. While he invokes and implies the deeper ontological perception, the more metaphorical approach wins out in the end. However, there is no reason why a further developed sociophysics may not yet involve more seriously the ontological approach.

4.0 Conclusions

The ambivalence of the great codifier of neoclassical economics, Paul Samuelson [30], regarding the role of entropy in economics is reflected in the duality of its role in econphysics. Entropy can be seen as an ontological driving force of economies, especially as one recognizes their profound reliance on ecosystems driven by solar energy, both current and stored, with this energy dissipating through the system as an inexorable working out of the Second Law of Thermodynamics as described by Georgescu-Roegen [11]. On the other hand we see the mathematics of entropy serving as a useful metaphorical tool for understanding various economic and social phenomena, even as these parts of the larger system may not in reality be driven by this inexorable process.

Despite this duality, we recognize the essential unity of the various versions of the concept of entropy. Any serious effort to use econophysics to understand the economic system will inevitably rely to some extent on some form of the concept of entropy.

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