**INCOMPLETENESS AND COMPLEXITY IN ECONOMIC THEORY**

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1. **Introduction: Chaitin’s Way**

 Kurt Gödel (1931) showed that the formalist program of David Hilbert was doomed by demonstrating that within a consistent formal system that a Peano axiomatized arithmetic can be carried out is incomplete. It will be able to generate statements that can neither be proved or disproved. Likewise, for such a system its consistency cannot be proved within the system itself. Central to the incompleteness theorem was the use of paradox based on self-referencing.[[1]](#footnote-1) “This statement is unprovable.” A great irony of the theorem is that Gödel used the system developed by Whitehead and Russell (1910-13) to avoid various logical paradoxes to generate his own paradox, thus hammering home the failure of the formalist program.

 This result soon inspired further important developments that would provide the foundation of computer science in the study of recursive functions (Church, 1936) Especially important would be Alan Turing’s conceptualization of a generalized form of computing device that could solve (or not solve) such functions, which would come to bear his name, the Turing machine (Turing, 1936). Indeed, realizing the possibility of not being able to prove a statement arising from a logical system, Turing extended this to the idea of not being able to compute a solution. The self-referencing of the Liar’s Paradox (“All Cretans are liars, and I am a Cretan.”) sets up the possibility for an infinite do-loop that never ends in a program as one goes back and forth between being a Cretan and not being a Cretan. A program is not computable because it never halts. This would become known as the *halting problem*, and it would be the key to the idea of computational complexity.

 While non-computability is clearly the extreme case of computational complexity, the question of defining such complexity and measuring its degree arose out of the work of Turing. Deriving from this case the idea that the length of a program could be the measure of such complexity gained prominence (Kolmogorov, 1965). It was into this discussion and study that Gregory Chaitin (1966) first became involved in these matters and began to influence the nature of the path these ideas followed. Inspired by quantum mechanics and recognizing that one can never reach infinity in a program, Chaitin discovered the number Ω, which is the probability of halting in a program. In the spirit of Gödel and Turing, while this number is definable, it is not itself computable. Study of this number and its implication that randomness is integral to the foundations of mathematics and computer science led to Chaitin’s (1987) development of highly influential *algorithmic information theory*.

 Chaitin has argued in various places that his randomization of mathematical logic provides a new way of relating underlying axioms to “mathematical facts,” which are never fully to be known. “Randomness = incompressibility” (Chaitin, 2002, p. 37). While the traditional notion is that a small set of finite axioms generates mathematical truths, algorithmic information theory implies going the other way to compress mathematical facts into a small set of axioms. With unproven assertions or programs that run for a very long time, estimating a probability of provability allows one to reach this irreducible relation between the two sides, a relation that at some level turns the mathematical facts into axioms.

 With influence pervasive on computer science as well as the foundations of mathematics regarding the nature of undecidability, Chaitin himself would branch out and expand his applications of these ideas to a broader array of disciplines. These would include philosophy and the arts (Chaitin, 2002), generalized complexity (Chaitin, 2008), evolutionary biology (Chaitin, 2012) as well as physics and economics (Chaitin et al., 2012). It can be argued that the strands of thought pulsing through these meditations have laid out a distinctive path of thought, a veritable *Chaitin’s Way*. This essay seeks to explore aspects of this multi-disciplinary, or even transdisciplinary, way that manifest themselves more broadly within complexity and economics. These include such matters as incompleteness and general equilibrium theory, how a constructivist mathematics changes economic theory, how this perspective links with complex reflexivity in economics, and how complex emergence arises within economics.

1. **Incompleteness and the Limits of Economic Theory**

 There has long been a pattern of mathematical ideas and issues in economic theory appearing in game theory before they have in general equilibrium theory (GET), with the application of fixed point theorems by Brouwer, initially done by von Neumann (1928. 1945) for both, followed by the Kakutani version initially by Nash (1951) for game theory to be followed by Arrow and Debreu (1954) for GET. So it has been the case with seriously considering problems arising from self-referencing and the incompleteness theorems.

 Thus probably the first recognition of such limits were for game theory and came from Michael Rabin (1957), who found an uncomputability result that there are games in which the player who can always win cannot do so in practice because it is impossible to supply her with effective instructions regarding how to play in order to win. This resembles Chaitin’s number Ω that is known to exist but cannot be computed. It can be approached by successive approximation, but not fully reached. It is already the case that even known game theory equilibria often involve mixed strategies that are already probabilisitic, with this uncomputability result simply adding another layer of stochasticity on top.

 Rabin’s result would be extended later to more general cases by Prasad (1991) and Tsuji et al. (1998) who showed a more general uncomputability of Nash equilibria. This would be followed by Koppl and Rosser (2002) who showed how self-referencing leads to uncomputability of best response functions in game theory. Their arguments rely on Cantor-style diagonal arguments of the sort used by Turing (1936). Their work drew on Binmore (1987) whose response to this problem was to propose a sophisticated form of Bayesian updating to approach solutions that, ironically, assume greater ignorance by the agents involved. However, this only holds in finite spaces and when the basis is continuous (Diaconis and Freedman, 1986). The paradoxes underlying these non-computabilities involve infinite regresses of agents knowing the other agents’ knowing of their own knowing of the other agents’ knowing (ad infinitum) of each other’s expected strategies. A famous example of this the Holmes-Moriarty problem and has a mixed strategy solution. The problem is how to get to that solution, with it not being possible by trying to use best response functions.

 Others began to understand that these problems can appear in a variety of areas of economics.[[2]](#footnote-2) Peter Albin (1982) observed that self-referencing and associated incompleteness with uncomputability arise in many problems where a micro decision solution depends on a macro result while at the same time the macro result depends on the micro decision solution. It can be thought that this might be solved by a giant handwaving assertion of general equilibrium, but in fact this is more a problem of aggregation, a problem that GET has seriously, as we shall see. A particular example Albin cites is the old problem of aggregation in capital theory that became entangled in the paradoxes of capital theory debates. In any case, the problem as posed by Joan Robinson (1953-54) was that on the one hand in order to aggregate capital one needs to know what the marginal product of capital is so as to aid in determining the discount rate for calculating present values, while at the same time one needs to know what the value of aggregate capital is before one can determine its marginal product. One is potentially stuck in an endless do-loop like Holmes and Moriarty trying to outguess each other as to whether to get off the train from Victoria Station at Canterbury or Dover.

 Which brings us to GET itself. Probably the first to raise issues based on self-referencing, incompleteness, and uncomputability emphasizing particularly Turing’s work was Alain Lewis (1986, 1992). His work relates to complexity in a curious way in that it reportedly heavily influenced Kenneth Arrow, playing a role in his advocacy of and support for establishing the Santa Fe Institute and getting economists involved with it from the start, which might not have happened otherwise as its main founders and inspiration came from physicists associated with the nearby Los Alamos nuclear physics laboratory. Lewis’s work in turn prepared the way for a more general result on uncomputability of Walrasian general equilibria due to Richter and Wong (1999). Their general result also applies to such simplifications of GET as dynamic stochastic general equilibrium (DSGE) models now so widely used in macroeconomic modeling.

 There are at least three problems of incompleteness with GET. One not directly tied to G$ö$del-Turing forms of incompleteness involves aggregation and is due to what is known as the SMD theorem (Sonnenschein, 1972; Mantel, 1974; Debreu, 1974). This theorem states that for a continuous, homogeneous of degree zero function, with Walras’ Law holding, there exists an economy with at least as many agents as goods for which the function is an aggregate demand function with prices bounded away from zero. This says effectively that there are no restrictions on aggregate demand functions if no restrictions are based on individual preferences, meaning any aggregate demand function based on a microfoundation is highly arbitrary. Stability and other useful characteristics cannot be guaranteed, with, indeed, the theory as classically conceived incomplete as a way of understanding economic phenomena. To add insult to injury, Kirman and Koch (1986) have shown that the SMD theorem still holds even with identical individual preferences and identical shares of income for agents.

 Unsurprisingly this is linked to incompleteness problems more clearly tied to the underlying Gödel theorems, very much in the spirit of Chaitin’s algorithmic information theory. This involves the problem of moving from an axiomatic system that allows for proving existence of general equilibrium to establishing uniqueness or stability. Ingrao and Israel (1991) have explained how this problem presented itself to Debreu (1959). Let **FS** be the formal system presented by Debreu (1959) based on axiomatic system **AX**. This system can be arithmetized by a gödelization procedure on each symbol, statement, and demonstration with natural numbers, thus allowing a direct application of the incompleteness theorem. When it becomes clear that certain outcomes such as stability cannot be established while relying on **AX**, then it can be expanded by adding new axioms, **A** \*, giving a new axiom system **AX + A\***, which in turn gives a new formal system, **FS\***. However, now according to the second part of the Gödel theorem this new system cannot be shown to be consistent from within itself. There is no end to this, and we are left with a probabilistic outcome as argued by Chaitin. Debreu himself was aware to a considerable extent of these problems limiting GET and how hard it was to get uniqueness and stability for his system even when it could prove existence, although his concerns generally focused on local results.

 To complete this section we consider issues more tied directly to computational complexity and unsolved problems whose probability of proof is characterized by a Chaitin Ω that is not computable. This involves a theorem on efficient markets by Maymin (2011) that involves skirting the boundaries of the still unsolved problem of the relation between P and NP complete problems. While most computer scientists and logicians believe that P does not equal NP, an argument first made by both Gödel and Nash independently and secretly in the 1950s, it remains one of the most important unproven hypotheses in computer science and thus subject to ongoing revision of its Chaitin probability as new efforts proceed.

The Maymin theorem posits that under certain conditions about information, markets are efficient if P = NP, which few believe is really the case. In a study that pushes the boundaries of our knowledge of these matters, da Costa and Doria (2016) consider the O’Donnell algorithm in connection with this theorem (O’Donnell, 1979). This algorithm is exponential and thus not P, although it involves an exponential function that is slowly growing and thus “almost P.” This holds if P < NP is provable for any theory strictly stronger than Primitive Recursive Arithmetic (PRA) while PRA itself cannot prove it. This is known as the *counterexample function to the P = NP problem.* The sorts of economics problems that appear to involve NP solvability problems include such computationally complex ones as the traveling salesman problem. Anyway, da Costa and Doria establish that under these conditions where the O’Donnell algorithm behaves as an “almost P” system, this then implies an outcome of “almost efficient markets.” This is a result that walks on the edge of the unknown, if not the unknowable.

1. **Incompleteness and Constructivist Economics**

Chaitin’s irreducible randomness of provability that attempts to generate a balance between axioms and mathematical facts not only has implications for economic theory, but has implications for mathematics itself. Nearly all current economic theory has been derived from a mathematics that accepts a standard, formalist, or “classic” set of axioms that fit the Hilbert approach and that is indeed used by most mathematicians themselves, this set of axioms having a virtue of providing easy ways of proving various theorems, even if many mathematicians retains doubts regarding the fundamental truth of them. This set of axioms is often labeled as being the Zermelo-Fraenkel-[Axiom of] Choice set, or ZFC. This set has long been viewed as easily consistent with the Peano arithmetic, and many mathematicians do not question this set and view worrying about their truth as an unfortunate distraction from getting at the work of proving theorems and developing new knowledge.

 Nevertheless, there has been an ongoing debate within mathematics over this set of axioms that dates back into the 19th century. For a period in the mid-20th century these dissenters were pushed back to far corners, but starting in the 1960s a new group suggested that the alternative approaches should be taken seriously (Kleene and Vesley, 1965; Bishop, 1967). As it is these alternative approaches, which go broadly under the label *constructivist*, emphasize relying on using computation to solve problems rather than questionable formalistic axioms that allow for neat and easy solutions. Thus the increasing power of modern computers, something that Chaitin has celebrated and emphasized, have probably helped encourage the revival of this constructivist approach.

 There are varying factions among mathematicians sympathetic to constructivism depending on which and how many of the standard ZFC set of axioms they question, but broadly there are two axioms and a principle that have been questioned by one or another of the constructivists. Probably the first axiom to be questioned starting in the late 19th century was the Axiom of Infinity. The fundamental argument is that infinity simply does not exist in the real world. It is the abstraction beyond abstractions, a complete fantasy. As it is, using sending variables asymptotically to infinity as a method in proving theorems is deeply embedded in large parts of classical mathematics. Of course such methods arguably remain in the world of the finite, and indeed what really upset the constructivist critics was the introduction by Cantor of levels of infinity, the “paradise” that Hilbert praised him for “bringing us [mathematicians] into.” Dumping infinity makes proving things more difficult and simply ends substantial discussions in mathematics, but constructivist critics argue that this keeps mathematicians more real and credible, and forces them to lay out computationally how to establish mathematical facts.

 The other axiom that has been questioned by many is the Axiom of Choice, which allows the relatively easy ordering of infinite sets. What is involved here may not be as profoundly metamathematically deep as the question of infinity, but in fact getting rid of the Axiom of Choice may be more damaging to classical mathematics and generate even more problems for proving various theorems. It may be less dramatic than the Axiom of Infinity, but the Axiom of Choice is probably a harder workhorse of the classical axiomatic structure, especially important in topology and parts of real analysis, even as its role remains less obvious and visible.[[3]](#footnote-3)

 Finally we come to a deep principle that dates back at least to Aristotle, if not earlier, and which also involves a deep perspective on the nature of truth. This is the *Law of the Excluded Middle*. Statements must be either true or false. It may seem that such approaches as Boolean algebra and fuzzy set theory, which allow for probabilistic statements, may look like they get around this sharp duality of truth versus falsehood. But in fact they do not do so as the very statement that “there is a 10 percent chance of a thunderstorm tomorrow” is itself arguably subject to this same harsh truth versus falsehood duality. Obviously this is a very philosophically profound position, and for classical mathematics it allows for the use of proof by contradiction, which has often been used in many of the most famous and neat theorems of mathematics, ranging from the original proof that the square root of two is not a rational number up to Cantor’s proof that there is no highest level of infinity and more.

 The particular branch of constructivist mathematics that emphasizes not accepting the Law of the Excluded Middle is known as *intuitionism*, and may be the most radical of the constructivist schools. The founder and most famous advocate of intuitionism was Luitzen Brouwer (1908), who also proved the first and most famous fixed point theorem. Ironically, Brouwer used non-intuitionist methods in that proof, with him only much later providing an alternative proof consistent with intuitionism (Brouwer, 1952). Brouwer essentially became the leader of the broader constructivist group in debates with Hilbert and others defending the formalist classical approach in the 1920s and 1930s, when it was widely perceived that the latter managed to get the upper hand (even as Gödel more profoundly undercut all of them in 1931). In any case, intuitionism has seen a revival (Bishop, 1967; Kleene, 1967), and its defenders do so on grounds of ultimate philosophical and ontological principles.

 Until quite recently these matters have rarely been even mentioned by economists with the occasional exception (Scarf, 1973). But indeed concern over these matters has become a focus of research among some economists, especially among those concerned with issues of computability of the sort Chaitin has long been concerned with (Velupillai, 2005; Bartholo et al. 2009; Rosser, 2012a). Unsurprisingly much of this has taken a critical turn by pointing out the non-constructivist elements in standard proofs of widely accepted theorems in economic theory. Thus Velupillai (2006, 2008) has pointed out how most proofs of the existence of general equilibrium rely upon fixed point theorems. However, both the Brouwer and Kakutani versions of this have traditionally relied upon using Sperner’s Lemma, which in turn depends on the Bolzano-Weierstrass Theorenm which in turn was proven using proof by contradiction.

 A common problem arising related to these concerns in computational economic models used for policy purposes has involved estimating approximate solutions using only the countably infinite set of rational numbers in programs. That this might lead to problems has been observed by Clower and Hewitt (1978) who posed a peculiar finance function for which rational number solutions are distinctly different from solutions for nearby irrational numbers. Velupillai (2005, p. 186) has noted that for such functions roundoff errors may lead to wildly divergent results that may have real world implications, such as the serious misfiring of a Patriot missile during the First Gulf War in Dhahran, Saudi Arabia that missed its target by 700 meters, thereby killing 28 soldiers in “friendly fire.”

 Some have argued that actual economic variables only take rational values so that therefore there should be no problem regarding this, Clower and Hewitt’s peculiar finance function aside. However, it would seem that at least some economic variables may take irrational values, with examples from real estate appearing, such as the diagonal length of a square piece of land. This sort of example may still not be too difficult to deal with ultimately given that polynomial roots are part of the set of algebraic numbers, which are also a countably infinite set like that of the rational numbers. One can still operate with a countably infinite set of numbers if all one adds to the set of algebraic numbers is a small finite set of the transcendental numbers with pi and e being the obvious candidates.

 There have been efforts made to overcome the problems of computing with real numbers (Blum et al., 1998). However, these efforts have generally not been accepted by the advocates of a constructivist approach to building up economic theory. It must be noted that while most of the literature in this area has taken a critical approach, there have been some efforts made attempting to develop just such a mathematical economics based on a fully constructivist approach (Velupillai, 2012).

1. **Reflexivity and Incompleteness**

 Closely related to self-referencing is the idea of *reflexivity*. This is a term with no agreed upon definition, and it has been used in a wide variety of ways (Lynch, 2000). It is derived from the Latin *reflectere*, which is usually translated to mean “bending back,” but can refer to “reflex” as in a knee jerking when tapped, not what is meant here, or more generally is linked to “reflection” as in an image being reflected, possibly back and forth many times as in the situation of two mirrors facing each other. This latter is more what the focus is here and more the type that is connected with self-referencing and all that implies. Someone who made that link strongly was Douglas Hofstadter (1979) in his *Gödel, Escher, Bach: An Eternal Golden Braid* as well as even more so in later works (Hofstadter, 2006). For Hofstadter, reflexivity is linked to the foundations of consciousness through what calls “strange loops” of indirect self-referencing, which he sees certain prints by Maurits C. Escher as highlighting, particularly his “Drawing Hands” and also his “Print Gallery,” with many commentators on reflexivity citing “Drawing Hands,” which shows two hands drawing each other (Rosser, 2019).[[4]](#footnote-4) Hofstadter argues that the foundation for his theory is the Incompleteness Theorem of Gödel, with its deep self-referencing, along with certain pieces by J.S. Bach, as well as these prints by Escher.

Chaitin himself appears not to have used this term in any of his discussions as near as I can tell. But others have used it in discussing some of his work, with the meaning clearly related to the paradoxes that can arise from self-referencing such as the Cretan Liar’s Paradox, an example cited by many who have written about reflexivity. Thus Mlynarski (2018, p. 1) argues “…that a formal system contains a sentence, which introduces reflexivity, does not imply that the same system does not contain a sentence or proof procedure which solves the problem.” This paper applies this to diagonal proofs such as that showing that the real numbers are not countably infinite and asserts that this shows a limit to the “Chaitin-Gödel Theorem” (Chaitin, 1974). So Chaitin may not have used the term (or at least not very much) in connection with these sorts of limits-to-mathematics discussions, but others are doing so in connection with his arguments about these matters.

The term has probably been most widely used, and with the greatest variety of meanings, in sociology (Lynch, 2000)). Its academic usage was initiated by prominent sociologist, Robert K. Merton (1938), who used it to pose the problem of sociologists thinking about how their studies and ruminations fit into the broader social framework, both in how they themselves are influenced by that framework in terms of biases and paradigms, but also in terms of how their studies and how they do their studies might reflect back to influence society as well. Among the sociologists the most radical uses of the concept involved sharp self-criticism wherein one deconstructs the paradigm and influences one is operating in to the point that one can barely do any analysis at all (Woolgar, 1991), with many complaining that this leads to a nihilistic dead end. The earliest usages of the term by economists followed this particular strand of analyzing how particular economists are operating within certain methodological frameworks and how they came to do so from broader societal influences and how their work may then reflect back to influence society, sometimes even through specific policies or even ways of gathering and reporting policy-relevant data (Hands, 2001; Davis and Klaes, 2003).

Merton (1948) would also use the idea to propose the idea of the *self-fulfilling prophecy*, an idea that has been widely applied in economics as with the concept of sunspot equilibria (Azariadis, 1981), with many seeing this as deriving originally from Keynes (1936, Chap. 12) and his analysis of financial market behavior based on the early twentieth century British newspaper beauty contests. In those contests newspapers would publish photos of young women and ask readers to rate them on their presumed beauty. The winner of such a contest was not the person who guessed which young woman was objectively the most beautiful, but rather which one received the most votes. This meant that a shrewd player of such a game was really trying to guess the guesses of the other players, with Keynes comparing this to financial markets where the underlying fundamental of an asset is less important for its market value than what investors think it is. This led Keynes even to note that this kind of reasoning can move to higher levels, trying to think what others think others think, and on to still higher levels in a potential infinite regress, a classic infinite reflection in a non-halting program. This beauty contest idea of Keynes has come to be viewed as a centerpiece of his philosophical view, implying ultimately not only reflexivity but complexity as well (Davis, 2016).

 Among the first to pick up on Keynes’s argument and apply it to self-fulfilling prophecies in financial markets and also bringing in reflexivity as relevant to this was George Soros (1987), who would later also argue that the analysis was part of complexity economics (Soros, 2013). Soros has long argued that thinking about this beauty contest-inspired version of reflexivity has been key to his own decision-making in financial markets. He sees it as explaining boom and bust cycles in markets as in the US housing bubble of the early 2000s, whose decline set off the Great Recession. He first got the term from being a student of Karl Popper’s in the 1950s (Popper, 1959), with Popper also an influence on Hayek (1967) in connection with these ideas (Caldwell, 2013). Thus the idea of reflexivity with links to arguments about incompleteness and infinite regresses associated with self-referencing have become highly influential among economists and financiers studying financial market dynamics and other related phenomena.

1. **Incompleteness, Complexity, and Emergence**

The concept of *emergence* has long been viewed as a central phenomenon in complex systems (Rosser, 2010, 2012b). Chaitin has at times written about it (Chaitin, 2008) usually stressing its role in such matters as creativity and the appearance of unexpected outcomes, with this aspect stressed also by Moore (1990). It is also deeply tied to elements of algorithmic information theory, particularly the matter of levels of computational complexity, with it associated with a transformational function that changes one state to another that is more computationally complex as defined by Chaitin and Kolmogorov (Lee, 2005). Discussing Chaitin’s ideas, Requardt (1991, p. 185) argues that emergence involves a “suitable extension of ‘algorithmic complexity’ to possibly non-recursive…structures in general, [and to] … possible existence of levels of complexity beyond, e.g. the human brain and the epistemological consequences thereof.”

Some argue that the idea of emergence goes all the way back to Aristotle’s discussion of situations where “the whole is greater than the sum of its parts,” although this has also been identified as being the core concept of *complexity* as opposed to mere “complicatedness” (Israel, 2005). However, many argue that the modern father of the emergence idea was John Stuart Mill (1843) who labeled it as an example of *heteroclinic laws*. For Mill these involved something more along Hegelian lines, a qualitative change arising from some interaction. His original example involved chemistry in particular how when one combines methane with oxygen one gets out carbon dioxide and water, two qualitatively different things. Certainly emergence involves this, although later many would add in the element of hierarchy, moving from one level to a presumably higher level, somehow defined, as well as a qualitative change.

Following Mill’s influence, Lewes (1875) coined the term “emergence” for these heteroclinic laws, and this led to a broad British intellectual movement known as *Emergentism* that emphasized essentially holistic approaches to a variety of subjects, including even theology. Probably the most important strand in this movement involved evolutionary theory, reaching a peak of development and influence in the 1920s with the work of Morgan (1923), whose great concern was explaining how more complex forms of life evolved out of simpler ones, how single-celled organisms developed into multi-celled ones with organs, how eventually consciousness would emerge, and so on. This program fell out of favor in the 1930s as the neo-Darwinian synthesis took hold that combined probability theory with genetics to emphasize a reductionist emphasis on individual genes (Gould, 2002).

However, these concerns would re-emerge later as these problems of qualitative change and leaps in hierarchical levels cried out for answering. Herbert Simon (1962) used such evolutionary emergence as a central example in his development of the idea of *hierarchical complexity* with Kaufmann (1993), Crutchfield (1994), and Bak (1996) tying this directly to computational processes. Boulding (1978) labeled these emergent hierarchical leaps to be forms of *anagenesis*, while Eigen and Schuster (1979) coined the *hypercycle* as the key to the emergence of multi-celled organisms out of single celled ones. Rosser (1992) would identify such transformations as being *hypercyclic morphogenesis*. Rosser et al. (1994) identified the moment of such a hierarchical leap as being the *anagenetic moment*.

Chaitin (2012) has also entered these waters with his *Proving Darwin*. Arguably he stays away from talking about emergence or even specifically about the specific nature of hierarchical or qualitative leaps, although they are implicit in his analysis. Rather he draws on the more reductionist neo-Darwinian synthesis approach, however placing it into the context of algorithmic information theory, unsurprisingly. It is a natural application, given the stochastic nature of mutations. He presents his famous number, Ω, as “the organism that evolves” (Chaitin, 2012, p. 65). He poses this as coming from *metabiology*, which emphasizes evolution as a computational process of increasing fitness in response to the random shocks of mutations. He invokes the *Busy Beaver problem* of Rado (1962), which involves naming numbers by first adding them then multiplying them then raising them to powers and to powers of powers, with this process leading to fitness that “increases faster than any computable function” (Chaitin, 2012, p. 42). While again he does not specifically highlight qualitative leaps, they are implicit in the jumps such a Busy Beaver function makes as it goes from one arithmetic operation to another, and he discusses such problems as Darwin’s own concern about how “half an eye is useless.” Clearly this is all very much up Chaitin’s alley.

 Let us return then to the idea of levels of hierarchy of computational complexity, with many proposing various systems for this. A common one involves four levels: linear programs, polynomial (P) ones, non-polynomial (NP) ones that still halt (although if one is running one must especially confront Chaitin’s Ω), and then the fully uncomputable ones that do not halt, the highest level of computational complexity. This has led many to identify other four level systems of complexity with this one, even if arguably this involves stretching things somewhat. Thus Mirowski (2007) introduced the idea of *markomata*, that markets are evolving forms of algorithms. He sees simple spot markets leading to the emergence of futures markets, which lead to the emergence of options markets, which in turn lead to higher level derivatives markets. He compares this four-level hierarchy to Chomsky’s (1959) four-level hierarchy of grammar: from regular language to context-free language to context-sensitive language to recursive enumerable language (that is, a Turing machine). He also sees it as comparable to Wolfram’s (1984) four-level hierarchy: from systems converging on a homogeneous state to ones that evolve towards simple separated or periodic structures to ones that evolve towards chaotic dynamical structures to ones that evolve to complex localized structures. Again, in all of these hierarchical systems, moving from one level to another is a matter of complex emergence.

 The link of all this to Gödel’s Incompleteness theorem goes back to the matter of how one resolves the appearance of incompleteness in a formal system. This involves moving to a higher level system with more axioms, arguably an emergent leap. But as noted above, one then faces the problem of determining whether ot not this system is consistent. There is no end to all this.

1. **Conclusion: Chaitin’s Transdisciplinary Way**

It should be quite clear by now that Chaitin’s Way is one that follows a transdisciplinary path despite his grounding in mathematics. Incompleteness implies the halting problem implies uncomputability and a hierarchy of levels of computability. This leads to his random dance between axioms and mathematical facts. But the existence of these levels and possibility of moving up and down them leads to complex emergence, with this phenomenon appearing across many disciplines from physics to chemistry to evolution to economics. This is a path that itself is subject to Chaitin’s own limit of its own uncomputable Ω. We do not know how or when it may halt, but the path is clearly one of ever-expanding knowledge and possibilities. May it proceed forward and further.

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1. For a more extending the theorem to a more generalized view of consistency, see Rosser (1936) in which the key is to show that for any sentence provable in the system a shorter proof of its negation can be found, this known as “Rosser’s trick” and also “the Rosser sentence.” See Kleene (1967) who provides a Turing machine based proof of Rosser’s version of the incompleteness theorem. [↑](#footnote-ref-1)
2. Some more general discussions of limits on economics due to self-referencing include Winrich (1984), Bookstaber (2017), and Landini et al. (2019). [↑](#footnote-ref-2)
3. On the one hand, Specker (1953) has shown that the ultimate Axiom of Choice that allows for the well ordering of all infinite sets is false, a sort of ultimate victory of this branch of constructivism. However, nonstandard analysis introduced by Abraham Robinson (1966) allows for infinite (and infinitesimal) real numbers, which allows for not needing the Axiom of Choice nearly as much for proving well-known theorems. [↑](#footnote-ref-3)
4. Examples of reflexivity in art are often thought to involve the *Droste Effect*, in which a work contains an image of itself within itself, clearly a matter of self-referencing. Among the earliest known examples is a painting by Giotto from 1320, *The Stefaneschi Triptych*, in which in the central panel Cardinal Stefaneschi is depicted kneeling before Saint Peter and presenting to him the triptych itself. Needless to say, even if they cease to be depicted after a finite sequence of images, such artworks exhibiting this Droste Effect imply an infinite regress of ever smaller images containing ever smaller images (Rosser, 2019). [↑](#footnote-ref-4)