**OBSERVATIONS ON COMPUTABILITY, UNCERTAINTY, AND TECHNOLOGY**

J. Barkley Rosser, Jr.

James Madison University

rosserjb@jmu.edu

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Abstract: Inspired by work of Stefano Zambelli on these topics, this paper the complex nature of the relation between technology and computability. This involves reconsidering the role of computational complexity in economics and then applying this to a particular formulation of the nature of technology as conceived within the Sraffian framework. A crucial element of this is to expand the concept of technique clusters. This allows for understanding that the set of possible techniques is of a higher cardinality of infinity than that of the points on a wage-profit frontier. This is associated with potentially deep discontinuities in production functions and a higher form of uncertainty involved in technological change and growth.

**Introduction**

 Stefano Zambelli is one of very few economists to have studied in depth issues relating computability of economic models (Zambelli, 2010, 2012) to technical change (Zambelli, 2004, 2005) and more indirectly to the nature of Sraffian economic models (Zambelli, 2018). A centerpiece of this panorama was concern with that most Keynesian of concepts, uncertainty. Uncertainty is most certainly uncomputable. Nevertheless economic agents act, and must do so on the basis of guesses and approximations, which may be computed, even as certainty always remains out of reach. This becomes even more salient when decisions are made to change technology. Outcomes cannot be known in advance. And the implications of changing technology in one sector for other sectors also can rarely be seen clearly in advance. Economies effectively stumble forward as their agents change how they operate without fully understanding what they are doing or where it will lead.

**Computational Complexity and Uncertainty**

 Underpinning concerns about computability of economic models as studied by Zambelli (2010) is the question of incompleteness as identified by Gödel (1931) and then applied by Church (1936) and Turing (1937). Self-referencing in Gödel’s Incompleteness Theorem plays a central role in generating unresolvable contradictions within systems stabling incompleteness. The classic form of this is the Cretan Liars Paradox, but there are others. Such paradoxes when appearing in a computer program introduce infinite do-loops that run back and forth without halting. This halting problem becomes the basis for uncomputability, with this holding for real numbers (Blum et al, 1998) as well as for the rational numbers that are actually used in computer models of the sort that concern Zambelli, although Zambelli (2011) would consider real computation as it occurs in analogue computers such as the Phillips Machine.

 It is generally accepted that the case where halting problems occur and programs cannot end represents the most extreme case of computational complexity. But computationally complex systems include many more cases, with a variety of competing ways to measure the degrees of such complexity, with many of these having been explicated by Velupillai (2000, 2005, 2009). Also it is well understood and widely accepted that computational complexity can underlie uncertainty.

 A deep thread in this matter involves relations between probability and information content. An important foundational concept has been that of *Shannon information*, which in turn derives from Shannon entropy (1948). If p(x) is the probability density function of a set of K states denoted by values of x, then the Shannon information content is given by

 SI(x) = ln(1/p(x)). (1)

This is regularly understood to be a useful representation of the number of bits in in an algorithm to compute the relevant associated code. It became the foundation for the highly influential formulation by Kolmogorov (1965) of *Kolmogorov complexity* (sometimes known as Kolmogorov entropy) measuring the minimum number of bits in an algorithm that does not prefix any other algorithm a(x) that a universal Turing machine would require to compute a binary string of information, x, given by

 K(x) – min la(x)l, (2)

with l l denoting the length of the algorithm in bits.

 This formulation has deep links to ones also by Solomonoff (1964) as well as by Chaitin (1966, 1987). Solomonoff was more also trying to link information with probability and drew especially on Keynes’s (1921) view of these matters. Chaitin would emphasize the importance of the minimum length description, tying it to his number Ω, the probability that a program will halt, which is the foundation of his broader work on *algorithmic complexity* (Chaitin, 1987; Rosser, 2020). This links back to the question of incompleteness, and Velupillai (2009, p. 72) has argued that Chaitin’s emphasis on incompleteness inappropriately narrows the conceptualization of computational complexity. In any case, these are all themselves uncomputable.

 A computable measure is that of stochastic complexity formulated by Rissanen (2005). This draws on these earlier measures and defines a likelihood function for a given structure of parametric density functions on k parameters and indexed data strings, x. From this a logarithmic measure can be constructed known as *stochastic complexity*, which can be interpreted as representing the shortest code length for the indexed data, x, that can be obtained with the given structure of parametric density functions or model class. While this measure has the virtue of being computable, it has probably been used less frequently than some of these other measures.

 Whichever of these measures of degrees of computational complexity one favors, it is widely viewed that four distinct levels of such complexity exist, even as there remain important unsolved problems regarding the nature of these levels (Wolfram, 1984; Mirowski, 2007). At the lowest level are linear systems, often described as not being computationally complex. They are generally solvable with algorithms of relatively short length. Qualitatively more complex are polynomial systems, or P. While usually involving substantially longer algorithms than linear ones, these are generally able to be solved by most current computers in a reasonable time. Generally accepted as the next qualitative level up are the non-polynomial systems, or NP. It remains unproven whether or not these are really different levels, with this problem initially identified independently in the 1950s by Gödel and Nash, although their work on this was highly classified for many decades. In any case, NP systems are solvable in finite time, but take far longer than P ones, with the Busy Beaver problem studied by Rado (1962) an example of much interest to Zambelli (2004, 2012). At the highest level of computational complexity are those that cannot be solved, that suffer from the halting problem, with some holding that only these are truly computationally complex.

 The most difficult unresolved problem in this is indeed the matter of whether or not P = NP, something the vast majority of those who consider the matter accept, but the proof of which has yet to be found. Curiously in Nash’s long-classified initial discussion of this matter he foresaw the promise and the problem. He considered the distinction to have relevance for problems of encryption and decryption and declared that he believed that indeed the two categories are distinct. But he noted that he was unable to prove that they were, and that this seemed to be a hard problem. Indeed it is hard enough that it remains unsolved, and its Chaitin probability, Ω, continues to change.

 Teasing us in regard to this problem are studies of the apparent boundaries between these probably distinct conditions. A curious example is the theorem of Maymin (2011) that shows that under certain conditions about information, markets are efficient if and only if P = NP, which, as already noted, very few believe. Following on this, da Costa and Doria (2016) have used the O’Donnell algorithm (O’Donnell, 1979) to study the functioning of markets in this likely boundary zone. This algorithm is exponential and thus not P, but one that is slowly growing and thus “almost P.” This holds for if P < NP is provable for any strictly stronger than Primitive Recursive Arithmetic (PRA) while PRA itself cannot prove it. This is known as the *counterexample function to the P = NP problem*. The sorts of economics problems that appear to involve NP solvability include the traveling salesman problem, well-known to be highly computationally complex. In any case, da Costa and Doria show that when the O’Donnell algorithm behaves as an “almost P” system, this implies an outcome of “almost efficient markets.” Arguably we are dealing here with an outcome that walks on the edge of the unknown, if not the unknowable.

 The links to the problem of uncertainty in various ways. One figure who deeply studied computational complexity in various of its forms and considered how this implies limits on the knowledge of humans and how they deal with this lack of certainty induced directly by such complexity was Herbert Simon (1955, 1957). While Simon emphasized more conventional issues such as people simply lacking information relevant to decisions they must make, he was also fully aware of the deeper sorts of problems we have been discussing here arising from the work of Church, Turing, and others. Simon saw these deep problems of uncomputability by humans themselves, not just computers, as the most fundamental reason why at best people must operate with bounded rationality. Ironically it was his study of these problems with humans that led him to study artificial intelligence in computers themselves (Simon, 1969). In any case, in a world of bounded rationality, one is dealing with an unavoidable degree of uncertainty. A more extended discussion of these issues is in Velupillai (2019).

 Of course one of the deepest students of the nature of uncertainty in the economy was Keynes (1921, 1936). A fundamental source of uncertainty for Keynes came from the nature of the interactions between people regarding their expectations and more particularly how their expectations depend on their expectations about what the expectations of others are. His classic statement of the problem came with his discussion of the beauty contest problem in Chapter 12 of *The General Theory* (Keynes, 1936). It has long been posed as fundamental to understanding financial markets, which was clearly Keynes’s main thrust when he wrote about it, but its application and implications go well beyond that context. To the extent one’s behavior depends on one’s expectation of what the expectations of others are, and their expectations depend on what one’s own expectations are, a situation of potentially self –referencing infinite regress is opened up that has the potential for being uncomputable and the basis for Keynes’s fundamental uncertainty (Rosser, 2004).

 About the same time, Oskar Morgenstern (1935) considered a similar problem in the form of the Holmes-Moriarty problem, which also has a potentially uncomputable infinite regress problem of reflexivity. In “The Final Problem” Moriarty is trying to catch Holmes as he travels from Victoria Station through Canterbury to Dover, hoping to catch him at one of those stations. Each contemplates the expectation of the other regarding where each will try to get off, and there is no end to levels of this set of expectations of expectations. Like Keynes’s beauty contest, it is an n-level game, but there is no end to how high n can go, which is completely arbitrary. It was Morgenstern’s concern about this problem that pushed him to collaborate with von Neumann in working on game theory (von Neumann and Morgenstern, 1944), with in effect a potential solution being found by means of mixed strategies: each agent simply tosses a coin.

 However, even as tossing a coin may be a Nash equilibrium solution, there still arises the problem from a computability perspective of how the agents come to this solution. It turns out that if one poses the problem as contemplating best reply functions, a hyper-rational Turing machine will find the problem to be uncomputable with the infinite regress problem reappearing to pose a classic self-referencing halting problem (Koppl and Rosser, 2002). Koppl and Rosser posed that the way out of this was to fall back on what Keynes proposed as the way out of the beauty contest problem as well, to rely on “animal spirits” to make the decision.

 Binmore (1987) considered the self-referencing problem and proposed that it be resolved using a form of Bayesian updating. This can be viewed as operating within a greater degree of ignorance, which has a similarity to the bounded rationality of Simon. But this form of updating will only operate properly under certain conditions as shown by Diaconis and Freedman (1986), notably that the basis is continuous and the problem is posed in a finite dimensional space. Otherwise there may be convergence to a cycle rather than a point, bouncing back and forth between two incorrect probabilities, again posing the problem of probability and information. An example might be a coin tossing that jumps back and forth between 1/3 and 2/3 without ever converging on 1/2., with Nyarko (1991) having studied such problems. Again, one may have to fall back on either an arbitrary heuristic a la Simon or more purely intuition a la the animal spirits of Keynes in order to act in the face of an uncomputably based uncertainty.

**Technology and Computabiity**

 Zambelli (2004, 2018) has shown concern about relations between technology and computability by analyzing characterizations of technology, based on Sraffa (1960). He has pushed the model beyond its usual limits by considering such variations as allowing multiple profit rates, although this arguably violates steady state equilibrium conditions usually assumed for such systems. The Sraffian framework allows for a perspective on technology and the choice of techniques that more clearly highlights deeper issues than the more standard neoclassical framework allows. Much of this arises from the avoidance of using aggregate capital concepts with a more precise characterization of technology. It must be recognized here that at least some neoclassical economists have accepted this at least in principle (Samuelson, 1966), but that recognition has generally been made by theorists, while empirical neoclassical modelers have tended to fall back on using Solow-style aggregate production functions with aggregate capital and simply ignoring these complications.

 Of course, much of the attention the Sraffian framework has received has been due to its usefulness for examining various capital theoretic paradoxes such as reswitching within it. These have been and are important questions, but the Sraffian framework also allows one to see into other aspects of capital theory that reveal issues of technical choice. Indeed, there are potentially problems related to computability that can be depicted within the framework not so easily depicted in more conventional models and frameworks.

 One matter the Sraffian framework allows to be revealed is that of the nature of continuity and discontinuity when looking at technique choice and the relations between different techniques along the wage-profit frontier of optimal techniques. In particular, that techniques may appear near each other on the wage-profit frontier does not mean they are near each other in any real sense, with ratios of factor inputs possibly being quite different. In short, while the wage-profit frontier may be continuous, indeed smooth, this does not mean that there is continuity across the techniques as one moves along the frontier. Realization of this has been known for a long time (Pasinetti, 1969; Garegnani, 1970; Harris, 1973). Thus, in particular from Pasinetti (1969) we have the following:

“For vicinity of any two technique on the scale of variation of the rate of profit does not imply closeness of the total values of their capital goods. It is therefore not true that, as the number of techniques becomes larger and larger, the differences in the values of capital goods per man and outputs per man of any two neighboring techniques necessarily becomes smaller and smaller. These differences might well remain quite large, no matter how infinitesimally near to each other two techniques are on the range of variation of the rate of profit. In other words, continuity in the variation of techniques as the rate of profit changes, does not imply continuity in the variation of the values of capital goods per man and of net outputs per man. This in fact seems to be one of the most important results emerging from the reswitching of techniques discussion. It seems to reveal capital theory as a field unsuitable to the application of calculus and infinitesimal analysis, and thus of marginal analysis.”

 Let us consider this from a generalized Sraffa-von Neumann approach that allows to describe more precisely techniques and their relations to the optimizing wage-profit frontier. We shall assume these all exhibit constant returns to scale. Let A be a non-negative n by m matrix of inputs and B a non-negative n by m matrix of outputs with n commodities and m activities, C be an n by 1 vector of final consumption, x is a positive n by 1 vector of activity levels, y and z are n by 1 vectors, p is a 1 by n positive price vector, δ is a 1 by n vector of depreciation rates, r is the rate of profit, w is the wage rate, A0 is a 1 by n vector of labor inputs, and L(t) = L(0)egt is the labor force at time t with g its growth rate, The set of possible techniques T = (α, β, . . . ) will include only viable techniques fulfilling y(α) = A(α)x(α) = B(α)x(α) – c(α) for at least one x(α).

 In equilibrium for technique α, the quantity system is given by

 (1 + g)A(α)x(α) ≤ B(α)x(α) - C(α), (3)

and the dual price system is given by

 wA0 + (1 + g)pA(α) > pB(α). (4)

The wage-profit curve is given by

 W = [A0(α)]-1p®[B(α) – (1 + r)A(α)]. (5)

Endpoints are given by respectively setting r and w = 0, with that for w = w\* given by

 w\* = [A0]-1pα(0)[B(α) – A(α)] (6)

with this equaling maximum steady state per capita consumption when g = 0, and that for r = r\*given by

 r\* = 1/λ\*(α), (7)

where λ\*(α) is the maximum eigenvalue of A(α)[I – δA(α)]-1, with r\*also being the maximum rate of balanced growth, the von Neumann ray.

 The wage-profit frontier (WPF) is the outer envelope of the set of wage-profit curves for all viable techniques in T, namely WPF = W(r) = (w, r^) such that for all r^ there exists a viable α in T for which

 P(α)[B(α) – (1 + r^)A(α)][A0]-1 ≥ p(β)[B(β) – (1 + r^)A(β)][A0]-1, (8)

for all viable β in T that are not α.

 Paradoxes of capital arise when there exist non-monotonic relations between r and steady state per capita consumption as r is varied for comparisons of steady state equilibria along the WPF, with reswitching being one such example, but not the only one. Rosser (1983) shows that the presence of such paradoxes, especially reswitching, where a given technique appears at distinct and separate points on the WPF, can lead to discontinuities in dynamic paths under certain circumstances. We also note that these non-monotonicties also occur between r and the ratio of the value of capital to labor along the WPF in a comparative manner, where that ratio is given by

 K/L = [(pC/L) – w]/(r – g). (9)

 While Pasinetti and others have seen that an apparently smooth WPF might well be enveloping an underlying set of possibly highly discontinuous relations, others have decided that if this is not the case then the non-monotonicities associated with various capital theoretic paradoxes can be avoided. In particular, such a claim was made by Yeager (1976). However, Rosser (1978) showed a counterexample based on an “eccentric reswitching” model in which variations in capital-labor ratios are smooth across the entire WPF, yet one finds a fundamental non-monotonicity, with the relationship between K/L and r reversing at the midpoint of the WPF and every technique except that appearing at the midpoint appears exactly twice on the WPF, once on each side of the midpoint, thus exhibiting reswitching. Garegnani (1970) also presented a similar case where there is smooth variation between techniques on the WPF surface even as the system exhibits non-monotonocities and reswitching.

 Let us now consider more thoroughly how varied the sets of techniques can be that can be represented within this Sraffian framework and how there might be a link to computability issues, opening up how this can complicate the problem of technique choice and the dynamic paths economies can take. For this we shall draw on and extend arguments first presented in Rosser (1991, Chap. 8). This can be viewed as drawing forth more fully the conceptualization presented by Pasinetti regarding the possibility of deep discontinuities even with a smooth WPF.

 So within the Sraffian tradition it has long been a practice to argue that techniques can be considered as appearing in a “book of blueprints.” Needless to say, this book may have an infinite number of pages, as there may be an infinite number of techniques. However, the cardinality of this infinite set of techniques may be up in the air. The image of a book with a set of successive discrete pages suggests that they may be counted, that the set may be of cardinality ﬡ 0. Assuming the continuum hypothesis holds then the next infinity up will have cardinality ﬡ1 and will be the continuum. Indeed, that is what we expect the cardinality of the set of points on the WPF to be, a continuum, which one associated with technique, one out of the book of blueprints that now has a continuum of pages.

 But this represents a narrowing of the possibilities involved here. As shown earlier, each possible technique is associated with a wage-profit curve, a function. As it turns out the set of functions, or the broader set of curved lines on a plane. The cardinality of such a set can in general be yet higher; it can be ﬡ2. It may be that for each point on the WPF there may be an infinite set of wage-profit curves associated with an infinite set of techniques passing through that point. Not only that, the cardinality of this set may not be merely countable or a continuum, but an even higher cardinality, that associated with the set of curved lines in a plane, although this may require broader technical possibilities..

 Here we return to the problem of possible non-computability. An economy may face a particular wage-profit pair. But this may not allow a computation to unequivocally select a technique. Not only may there be more than one possible technique, there may be a higher order infinite set of techniques. It may be that there are even sub-sets within these infinite techniques, all associated with a single point on the WPF. The techniques may be separated into *technique clusters*. These may include infinite sets of techniques that only slightly differ from one another in a physical sense, possibly representable as sets of neoclassical production functions or relations resembling those. The idea of technique clusters has been used in microeconomics, such as to compare TVs and VCRs. But here we are talking about technologies describing entire economies. In any case, through a given point on a WPF, there may be multiple technique clusters passing, each exhibiting reasonably continuous variations within themselves, but possibly highly discontinuous changes across each other, with their possibly being an infinite set of these clusters with a cardinality higher than the continuum.

 This is not precisely analogous to computational complexity, but it resembles it in the sense that some problems of computational complexity as noted in the previous section can involve disjunctures between different levels of such complexity, such as trying to solve an NP system using a P level algorithm. We do not think that it can be done, even if that has not been definitively proven yet.

 We present below in Figure 1 a WPF with only four different wage-profit curves, each associated with a very different technique. They exhibit the basic idea posed by Pasinetti, where we see techniques appearing near each other on the WPF, but exhibiting very different shapes and characteristics implying deep discontinuities between each other.

Figure i: Continuous Technologies with Discontinuous Profit-Capital Intensities

 This brings us back to the problem of how to decide what to do in the face of such profound uncertainty. These are decisions that are not properly computable and thus must be made on the basis of other means, whether some form of Simonian bounded rationality or more intuitively along the lines of Keynesian animal spirits. We close this discussion by noting a historically inspired comment by Joan Robinson (1975, pp, 38-39):

“When investment us being planned, whether for a single plant or for a socialist economy the choice of technique is made by groping amongst incomplete information. There are no read-blue-printed techniques to choose from. When the technique to be installed is designed to give a higher output per man than that formerly in use, it must be a recent innovation or an adaptation from one already known in the broad, not in detail. The blueprints are drawn when the technique is chosen, and it will rarely turn out, after the event, that exactly the best possible choice was made.

There is no such phenomenon in real life as accumulation taking place in a given state of technical knowledge. The idea was introduced into economic theory only to give a meaning to the concept of the marginal productivity of capital, just as the pseudo-production function was constructed in order to show that it has no meaning.”

**Conclusions**

 Computational complexity profoundly implies uncertainty as its level rises. This is intimately tied to the deep considerations of links between information and probability as studied by such figures as Shannon, Solomonoff, Kolmogorov, Chaitin, and Rissanen. Chaitin’s probability of a program halting encapsulates this, even as it possibly oversimplifies it. And the fullest uncertainty arises when the highest level of computational complexity is in place when reflexive self-referencing implies infinite do-loops and the presence of a halting problem. This problem, deeply associated with Gödelian incompleteness as realized by Church and Turing and others, forecloses any strictly computational solution of problems when it appears. But even short of that, when systems are NP as with the busy beaver problem, solutions may be extremely difficult to achieve. This is why Nash viewed the use of such systems as useful for encryption problems, although we know that some NP problems may not be so difficult, with such systems implying possibly even better behaved economies. In any case, when uncertainty due to computational complexity becomes too great, agents must rely on either some sort of procedural rationality along lines suggested by Simon or in more difficult cases fall back on more intuitive animal spirits as suggested by Keynes.

 Somewhat similar issues can arise in the problem of choosing techniques. The Sraffian framework offers a perspective in which examining available techniques as they array along a wage-profit frontier can highlight comparisons. But this also reveals that simplistic assumptions held in much of neoclassical analysis may not hold. Non-monotonic relations between capital-labor ratios and profit rates may occur that can imply possible erratic dynamic behavior. Apparently smooth systems with techniques lying infinitesimally close to each other on the frontier may represent highly different techniques revealing the frontier to be a surface hiding profound discontinuities. Indeed, the set of possible techniques may well be of a cardinality of infinity higher than the continuum, making computability of selecting a technique highly complex. The matter of technical choice may come to possess the sorts of problems discussed in the earlier portion of this paper. Thus the problem of technical choice may also ultimately depend not only possibly on some heuristic approach, but ultimately on reliance in the end on intuitive animal spirits of a Keynesian sort.

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