‘Consistency and Incompleteness in General Equilibrium Theory

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Abstract We consider the implications for general equilibrium theory of the problems of consistency and completeness as shown in the Gödel-Rosser theorems of the 1930s. That a sufficiently rigorous consistent system is incomplete poses serious problems for dealing with unresolved problems in a fully formal system such as general equilibrium theory. We review the underlying mathematical issues and apply them to this problem for general equilibrium theory. We also consider alternative approaches such as agent-based modeling founded on empirically estimated behavioral assumptions for agents that may allow for a better way to model non-equilibrium evolutionary economic dynamics.

Keywords Axiomatic Formal System, Consistency, Incompleteness, Gödel-Rosser's Theorems, General Equilibrium Theory, Agent-Based Modeling

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1. Introduction

Economic theory and empirical data do not intersect. As Kaldor summarized (1972, p. 1239, footnote 1): “In economics, observations which contradict the basic hypothesis of prevailing theory are generally ignored: the 'theorist' and 'empiricist' operate in two isolated compartments and the challenge of anomalous observations is ignored by the theorist as something that could be taken into account in the stage of “second approximation” without affecting the basic hypothesis”. If theoretical predictions do not match with facts this is not a proof of inconsistency and it does not undermine the credibility of the theory because there is no possibility of falsification.

A great deal of emphasis has been posed on the capability of developing scientifically consistent theories instead of ~~their~~ matching them with the data. ~~[SL, the reviewer is not too much convinced that the “as if” principle undermines the methodology, we should develop more clearly this point: The “as if” methodology]~~ ~~(Friedman 1953), which dominated, and still rules, the consensus view, allowed economists to mis-interpret axioms (i.e. revealed or indisputable truths) as if they were hypotheses (i.e. empirically testable assumptions).~~ The desire ~~o~~to~~f~~ scientifically provide~~ing~~ instances is so invasive that the analysis of real facts is replaced with a true axiomatics - often presented as simplifying hypotheses or assumptions.[4] If a hypothesis or an assumption is introduced in an economic model without any empirical testing these actually are axioms.[5] [6]

We will argue in this paper that the adoption of such a methodology makes (i) the mainstream approach non falsifiable (Section 3); (ii) the General Equilibrium Theory (GET) -also according to its modern version (DSGE, Dynamic Stochastic General Equilibrium)- is subject to the Gödel-Rosser's theorems, which make it unusable (Sections 4) and that (iii) a way out from the axiomatic trap is opening the formal system to testable hypotheses or simulation (Section 5). The

4 Antonio Palestrini nicely summarizes the problem with the expression: *in economics one– says “let's assume”, meaning actually “let's axiom”*.

5 As an example, think of the representative agent paradigm. There is no consensus on what the representative agent is or should represent (Jerison 1984, 2006; Kirman 1992; Hartley 1997), nor have there been attempts to prove or disprove its real existence. Statements like “the representative household maximizes utility...” is not a hypothesis nor an assumption, but an axiom because it undoubtedly states how a household behaves.

6 This is not a subtlety for a discipline which is grounded on mathematical economics as a formal system. Formally, hypotheses and assumptions are but temporary truths which are involved to proceed with the modeling, in advance of testing for them. Axioms are statements which do not follow from other statements as their premises and do not need of being proved, either because they are revealed or self-evident truths or because they are indispensable to analytically develop a chain of consistent deductions. As reported in Lolli (2002), Dedekind, the early father of *logicism* together with Gottlob Frege and of the theory of sets together with Cantor, was used to saying that nothing in Science which is provable should be accepted without proof.

paradigm of Walras was based on analogy with the classical physics inspired by~~to~~ the mechanicism of Poinsot (1842) but it was not sufficient to solve the fundamental problem in the general equilibrium framework~~,~~ that is proving the existence of the equilibrium. The major part of the im~~un~~balance between theoretical and applied economics is due to the axiomatization of the discipline according to the paradigm of Hilbert introduced in economics by von Neumann (1937) and later~~ly~~ by~~since~~ Arrow and Debreu (1954) and Debreu (1959). This is because, by the economists, the axiomatic mathematization of a theory is believed to be a sort of "scientific accreditation" which guarantees consistency of the theories, even when there is no empirical validation. The axiomatic mathematical form of economics is ~~so~~ privileged to substance to assert propositions~~take its place~~, without possibility of falsification, similar~~ly~~ to theology but different~~ly~~ from other disciplines. This worrying trend comes not without implications. No plausible critique is possible from the outside from the empirical evidence: the methodology strictly follows the late procedure of adding new ~~[SL, this might be the place do develop the analogy as asked by the reviewer, isn't it?:~~ *epicycles*~~]~~ to fit the data.[7] If economics is not falsifiable, the only possible critique is to~~could~~ prove the logical weaknesses of the theoretical approach (Kirman 1989). In the following we show how the Gödel-Rosser's theorems undermine the GET-DSGE in terms of coherence and incompleteness and offer an alternative approach: ABM (agent-based modeling) as a tool for dealing with complexity and falsifiability.

Mathematical economics from Walras *Elements* (1874) to the end of the 30's of the last century was basically an analogical representation of economic facts with the classical mechanics models of physics (Ingrao ~~e~~and~~d~~ Israel 1991). This mathematization of economics consists in changing the language from ordinary (e.g. English) to artificial (mathematics) and by finding analogies between economic and physical phenomena to formulate economic theories. The mathematical economics between the 40's and the 50's -von Neumann (1937), Samuelson (1947), Arrow and Debreu (1954) and Debreu (1959)- is the formalization of the methods to analyze economy. Formalization of economics consists in the mathematical specification of economic problems. Von Neumann (1937) formulated the equilibrium problem replacing the analogy with physics with a mathematical analogy (Gloria-Palermo 2010). In this mathematical economics of second generation the problem ceases to be economic to become a mathematical problem (Blaug, 2003).[8]

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[7] As it became clearer and clearer during the late Middle Ages that the Ptolemaic theory of the sun going around the earth was not predicting celestial mechanicsl dynamics well, its advocates kept adding small "epicycles" to the predicted main cycles to get it to explain the data.

8 By following the formalistic approach of Hilbert economic terms become pure syntactic symbols to be managed with the principles of mathematics. In all this process the core notion is that of *formal system* and economics

The switch from the first to the second generation of mathematical economics is a consequence of what happened in mathematics in the 20-30's: the foundational crisis, the controversies about ~~on the~~ set theory and ~~on~~ the notion of infinity, the debate among the competing foundational schools of mathematics, the formalistic Hilbert~~'s~~ program, the Gödel~~'s~~ theorems, and their consequences.

2 The Formalistic School of Mathematics Influence in Economics

~~To exit the crisis of fundamentals in mathematics, at the end of the 20's, Hilbert suggested to concentrate on the methods. Theories should be founded on a restricted and complete set of axioms to prove to be coherent:~~ [SL, ~~could we cut this?: the proof of coherence of higher order theories is to be reduced to the proof of consistency of lower order theories and, as a consequence, the coherence of the whole mathematics reduces to the proof of coherence of arithmetic~~]~~. Moreover, any mathematical theory should be reduced to its formal system in such a way that statements in the theory should be deducible (i.e. provable) by means of its axioms, within a finite number of inferential deductions which recursively apply to the axioms' set, which is therefore to be decidable or computable:~~[9] ~~if this is so then the theory is said recursively axiomatizable. Once this was done, epistemological legitimacy would be just a matter of consistency.~~

David Hilbert applied his approach to problems from the mathematized sciences, geometry and, above all, to the questions arisen from the crisis of the fundamentals, about which he wrote a

becomes a mathematized science. A formal system is a conceptual framework to formalize knowledge upon a decidable set of axioms as principles of deduction and inferential rules for deducing theorems of a formal axiomatic theory. Every statement in the formal system is developed according to a language which owns its alphabet of typographic symbols to write words according to grammatical rules which, according to syntactic rules, generate strings to be combined into expressions called well formed formuli (*wff*) or statements. Therefore, the grammar~~ties~~ generates words from an alphabet, syntax generates statements and semantics gives meaning to expressions. Every formal system owns a set of axioms, i.e. a structure of *wff* equipped with an effective method, that is an algorithm which, in a finite number of steps, returns an outcome or nothing. If the algorithm may return a *yes-no* answer about a question then the question is said to be *decidable*, if the answer may be only *yes* or *nothing* then the question is said to be *semi-decidable*: notice that the question itself can be a statement in the formal system. If the question is about a formula or a statement for it to be an axiom, then the effective method must prove that it does not depend on other axioms (or any other expression) as its premises. Notice that deciding whether a formula is an axiom or not does not require truthfulness or falsehood of its content, this is because an axiom is a statement accepted as an evident or revealed truth. From here onward only decidable sets of axioms are considered. A theory, as a set of theorems deduced from a (decidable) set of axioms, is recursively axiomatizable because, while developing a theory (a) the effective method operates on any *wff* stated upon the set of axioms, (b) which retrieves itself by means of the effective method to add or not a new formula in the theory and (c) the cognitive contents of a theory is enclosed in theorems deduced from axioms. Finally, inferential rules are instructions operating on *wff* in the formal system to develop demonstrations ending with a theorem.

~~9 A set is recursive, computable or decidable if there exists an algorithm (i.e. an effective method) which halts in a finite number of steps once the characteristic function of the set returns 1 (for *yes*) or 0 (for *no*) to decide whether if an element in the universe of the discourse is a member of the set or not. A set is semi-decidable if the effective method can decide only if an element is a member of the set.~~

program.10 The approach was based on *finitistic* methods, *axiomatics* and *formalism* without renouncing Cantor's infinity.[11] Symbols should be pure syntactic forms, and their number had to be finite. ~~[SL, shouldn't it be “formulæ” instead of Formuli]~~ should be syntactically well formed by means of a finite set of syntactic rules. Without resorting to intuition or experience, among such ~~[SL,~~ formuli~~]~~ a finite and restricted set of axioms should have been isolated to give proof of their coherence. Reasoning should have been correct in the sense that from premises, with a finite number of inferential rules to be applied a finite number of times, all the theorems should have been recursively deduced from axioms. Every element should have been rigorously explicated~~-~~ to avoid ambiguities. Also, the *Principle of Non-Contradiction* (no statement can be both true and false), the *Principle of Bivalence* (every proposition is either true or false), and the *Principle of the Excluded Middle* (for any proposition either it is true or its negation is true) are indispensable. ~~[SL, I would suggest formalizing these principles in a short appendix, this would meet the reviewer request: do you agree with this?]~~

The more controversial argument was that of infinity. Mathematized sciences and pure mathematics, heirs of the Aristotelian logic, did not consider reasonable the *actual* infinity. But, without it, it is difficult doing mathematics, for instance because the notion of limit would have been impossible to use.[12] Moreover, Cantor showed that a more extended number of theorems became provable in set theory by using ~~the~~ actual infinity (Boylan and O'Gorman 2007). ~~Therefore, the Cantor's interpretation of infinity should have been preserved [SL, maybe we should cut this: because Hilbert aimed at mathematizing all the sciences and give proof of their coherence and of mathematics as well].~~ Hilbert understood that actual infinity should ~~have~~ be~~en~~ preserved in the finitistic method (which allows only finite numberss to exist) by considering it as an ideal, a tool to extend demonstration capabilities as long as it would not bring the reasoning to some inconsistency. Finitism, formalism, recursive axiomatization and the Cantorian infinity are the elements which,

10 For a clear and rigorous historical description of the Hilbert's program see the Stanford Encyclopedia of Philosophy at https://plato.stanford.edu/entries/hilbert-program (retrieved in February 2017).

11 In the same year Walras published the *Elements of Pure Economics*, 1874, Cantor proved that some not finite sets can have different cardinalities, for example the cardinality of real numbers is not the same as that of natural numbers. In 1897 Cantor completed his theory of sets. The big impact was that he brought ~~the~~ *actual* infinity to the attention of mathematicians who, on the basis of Aristotelian logic, had ~~ever~~ excluded it as unreasonable, as absurd (Galileo) or a “way of saying” (Gauss) with respect to the *potential* infinity. Not without controversies the Cantor theory of sets was finally accepted: such theory proves that the power set ***P(N)*** of the natural numbers ***N*** has a cardinality which is greater that the cardinality of ***N***. Cantor proved many things but one is of great interest: ~~it~~here does not exist a cardinal number which is greater than every other cardinal number and that the set of not finite sets is itself not finite but it is the greatest of all the not finite sets it enumerates. The actual infinity was presented in the temple of mathematics.

12 The strict avoidance of infinity of any sorts is advocated by the constructivist school of mathematics that began with Kronecker in the late 1800s, who specifically criticized Cantor sharply. Moreover, the notion of limit was ~~lately~~ introduced by Weierstrass to overcome the ambiguity about the *infinitesimal* left by Leibniz and Newton: curiously enough, Weierstrass himself believed in ~~the~~ potential infinity.

according to Hilbert, could~~ have~~ mak~~d~~e possible deciding the truth or falsity of any mathematical statement, solving any possible question in the crisis of the fundamentals and lead to a complete formalization of mathematics.

~~[SL, I am not too much prone at cutting the following and, actually, I am not able to write the same things in a shorter way, in case you think the following is too lengthy, could you please provide a synthesis? Note that footnotes involved are relevant, I think. In case we would not remove the following we should give a convincing rebuttal to the reviewer: ~~Kurt ~~The young ~~Gödel appears i~~n this scenario by proving the *Completeness Theorem* in 1929, to be published the following year, as preparatory for the great discovery he would have pronounced soon as "un coup de théâtre". I~~n 1930, in Köningsberg, where there was a congress on the fundamentals of mathematics. The triad of the foundational schools were represented: *logicism*, *intuitionism* and *formalism*. Rudolf Carnap represented logicism,[13] Arnold ~~Unfortunately none of their fathers were there: Frege, who died in 1925, was the justified absent, Brouwer was absent because he could not stand Hilbert, who reported sick, only the ensigns were present. ~~Heyting represented intuitionism, and John von Neumann represented~~-~~ formalism~~ and Carnap logicism~~. The initial conclusion was that logicism's contribution was limited and it was agreed that formalism had won over~~n~~ intuitionsm.

Then ~~It then happened that ~~Gödel politely asked to speak and summarized his just-proven *incompleteness theorems* as~~. As he was used to, he said few words to synthesize the contents of the two incompleteness theorems, which will be discussed in the following. The triumph of formalism lasted only few minutes. Gödel said:~~ "If we stick to these fail-safe methods there will always be truthful conjectures that cannot be proved and mathematical problems that can never be solved. We can adopt the methods of sure reasoning, but in this way there will be issues that we will be unable to solve. Or we can have the potential capability to solve all the problems, but without the certainty of having solved them properly. We will never be certain of the methods and at the same time have the capability to solve all the problems".14

It is said that when Hilbert phoned to know what the conclusion of the congress was, von Neumann said that he had ~~a ~~good and ~~a ~~bad news. The good news was that formalism won over intuitionism,[15] the bad news~~one~~ was that "a certain Gödel claims that we lost too": formalism won the battle but, unexpectedly, not the war].

[13] Founded by Frege, logicism tried to reduce mathematics to being a subset of logic, with the term initially coined by Bertrand Russell but then spread in the late 1920s by Carnap.

14 Translation by the authors from Piñero (2014).

15 Founded by Heyting's mentor, L.E.J. Brower, famous for his fixed point theorem, ~~I~~intuitionism can be seen as a part of constructivism, which came from Kronecker's work. Fundamentally intuitionism rejects the law of the excluded middle. ~~and ~~W~~w~~hile it was viewed at this conference that it was defeated by formalism, many have since further developed and advocated~~-~~ the intuitionist approach (Kleene and Vesley, 1965).

If ~~To the end of this note we are interested on that~~ formalism ~~which~~ considers the *Principle of Non-Contradiction* as indispensable to the construction of ~~the~~ formal systems and ~~to develop~~ axiomatic formal theories, it adopts a language and recursive enumerable logic that~~and~~ involves correct reasoning at the syntactic level, because it is interested in provability of statements, not depending on the truthfulness or falsehood of their content or on the relationship with reality. To the formalist ~~the~~ reality is only an inspiration to develop a mathematical theory. Mathematics is not a method for the investigation of reality ~~-~~since a mathematical theory exists and it is valid because of its internal coherence even if separated from the facts that~~which~~ inspired it. Formalization is the mathematical development of a theory as grounded on its formal system.

This basically was ~~the~~ Hilbert's formalism, who aimed at axiomatizing all~~the whole~~ of mathematics and proving its non-contradictory *coherence*. T~~Unfortunately t~~his was found to be impossible because Hilbert's program was done in by ~~the~~ Gödel's theorems of incompleteness. Nevertheless, formalism resisted -and ~~it~~ still exists.~~:~~ V~~v~~on Neumann was the first to understand the implications ~~and the consequences~~ of ~~the~~ Gödel's theorems and he applied the formalistic paradigm in economics. Debreu[16] was one of the most distinguished formalists ~~of the modern era~~ who conceived formalism as the science of formal methods in economics.[17]

With~~Since~~ von Neumann (1937) and Arrow and Debreu (1954) ~~the degree of~~ axiomatic formalization is pushed to the limits: Blaug (2003) stated that “what was an economic problem […] has been transformed into a mathematical problem”, such that the forms take the place of the substance (Katouzian 1980). However, it is with Debreu (1959) that axiomatization in economics reached the highest level of rigor, primitive economic concepts had been reduced to purely syntactic symbols and all ~~the~~ interest in the theory was ~~a~~ mathematical interest. Debreu ~~himself~~ writes “Allegiance of rigor dictates the axiomatic form of the analysis where the theory, in its strict sense, is logically entirely disconnected from its interpretation” (Debreu 1959).

To Debreu a formal theory is ~~to be~~ rigorously constructed on an~~the~~ axiomatic basis without interpretation;~~,~~ this comes after[18] because ~~, at least in principle,~~ for the formalist the choice of the

16 Weintrau~~tau~~rb (2002) claims that Debreu was only interested in axiomatics while Boylan and O'Gorman (2007) explain that finitism was relevant as well for him.

17 In modern formalism everything resolves around the notion of demonstration, as well as the need of emptying statements of all semantic content to develop theories at the syntactic level. Axiomatics and finitism are still present but some distinctive traits had been relaxed (Lolli 2002): some post-Hilbert streams of formalism do not consider the *Principle of Non-Contradiction* as indispensable because ~~, as it will be discussed,~~ Gödel proved that coherence cannot be proved from the inside of the formal system. What matters is that formal theories are decidable, without possibility of syntactic errors but, at the same time, without the ambition of providing any absolute certainty about results.

18 Debrue’s Bourbakist mentor Cartan (1943) wrote: “The miracle of science is that we can build an abstract mathematics that can be lately effectively applied to the laws of nature”. Translation by the authors of the quotation from Cartan (1943) reported in the Italian edition of Ingrao and Israel (1991) on page 269.

axiomatic set can~~ould even~~ be arbitrary. ~~To the pure economic formalist it is irrelevant how much of economic is developed in the axiomatic formalization of an economic theory. And Debreu was an extreme formalist:~~ "The theory of value is treated [here] with the standard of rigor of the contemporary formalist school of mathematics" (Debreu 1959). The emphasis on "contemporary formalist" is unavoidable: Ingrao and Israel (1991) explain that this refers to *Bourbakism*, a post-Hilbert stream of formalism that~~which~~ aimed at achieving~~bring to~~ the extreme level ~~the approach~~ of Hilbert's approach. As a critique of~~to~~ the mainstream economic approach, ~~in terms of the present paper~~ this can be written as follows: above all is the form, the theory and its coherence, after that it will exist as place and ~~a~~ time in which the theory is correct~~, with an appropriate interpretation of phenomena in their environment~~. However, even for the most extreme formalist like Debreu, a formal economic theory does not appear out of thin air, it takes at least a propaedeutic event as inspiration: ~~that was~~ the economic general equilibrium of Walras.

All such reasoning explains the axiomatic formal level of Debreu's mathematical economics, but there is one more arguments that~~which~~ tunes Debreu with Hilbert's method. Hilbert~~–~~ was interested in Cantor's infinity to improve the capability of formal demonstration. As von Neumann also Debreu was unsatisfied by the analogy between the early mathematical economics and physics, which was based on differential calculus, but Debreu was not prone to renounce his Walrasian origin, although Walras (1874) had been inspired by the mechanicism of Poinsot (1842). By means of Hilbert's axiomatic formalism Debreu overcame that dissatisfaction while maintaining a Walrasian point of view but with a specific trait: to Walras the problem was an economic problem while to Debreu it was a mathematical problem. Therefore, in the ~~very~~ spirit of Hilbert, to solve his problem Debreu agreed to the Cantorian infinity in the axiomatization of prices and action plans on the space $\boldsymbol{R}^l$,[19] thus gaining the use of ~~with all the advantages that this could give in terms~~ algebraic topology, convex analysis, and fixed point theorems.

T~~Here, t~~hence it~~,~~ appears that Walrasian equilibrium gives the inspiration. The axiomatic formalism induces the isolation of the needed axioms to proceed syntactically, free from any economic semantics: that is, by correctly specifying and solving a particular mathematical problem, regardless of any economic content. The formalized primitive concepts do not need ~~of~~ any empirical test because the mathematical structure they define is always syntactically valid.

[19] While Debreu allowed the use of Cantorian infinity in proofs, his (and Arrow's) version of GET assumed a finite space of agents and commodities. These limits would later be expanded. Aumann (1964) would allow for a continuum of agents to directly prove equivalence of game theoretic core and competitive equilibria for many agents. Bewley (1972) assumed a continuity-imposing Mackey topology to allow for considering an infinite set of commodities. Fishburn (1970) and Kirman and Sondermann (1972) considered Arrow's impossibility theorem with infinite agents, finding that results at the finite level did not carry over straightforwardly to the infinite level. Sometimes assuming actual infinity strengthens conventional results; sometimes it does not.

Moreover, any possible ambiguity of the ordinary language of economics is removed to rigorously write theorems, as explicit as their premises, at the end of correct reasoning. Hilbertian formalism using~~with inclination to~~ the Cantorian infinity allows to prove, in terms of algebraic topology and set~~s~~ theory on $\boldsymbol{R}^l$, the existence of equilibrium. With such a method, within the Walrasian tradition, Debreu proved the existence of equilibrium but he never managed to prove its uniqueness or stability. On the other hand he gave a ~~substantial and almost~~ definitive contribution to the ~~so-called~~ *SMD Theorem*, as discussed below~~which will be discussed in a following section~~.

According to ~~the~~ mainstream economics, which evolved from Debreu's axiomatic economics, what matters is that a theory is a syntactically coherent mathematical structure with respect to a formal system upon which it is developed: syntactic coherence is then assumed as a value of truth. It will be the interpretation of axioms, in terms of a fact in its environment, ~~,~~ which~~that~~ will give semantic coherence a value of correctness, a truth ~~that is~~ never absolute but only relative, sufficiently generic but not general ~~since, according to the "as if" principle, any hypothesis can be assumed as long as it is satisfactory enough to describe reality~~. If something is ~~then~~ missing, e.g. that the implications of the theory do not reconcile with facts, it is not a problem: "the world as we see it" is a rigorous approximation of "the world as it is". Therefore, if what is missing is related to the limit of our ignorance, it only takes to add it to the model, maybe as a residual: otherwise, it takes finding a place and~~or waiting for a~~ time ~~at~~ whereich the necessary conditions hold~~realize~~ for the implications of the theory to match with facts.

3 Coherence and Incompleteness Relevance to Economics

According to the *Principle of Non-Contradiction*, a formal system is coherent if none of its statements contradicts another one. According to the *Principle of the Excluded Middle*,[20] together with the *Principle of Bivalence*,[21]~~,~~ regardless of its intrinsic truthfulness or falsehood, a statement is to be decidable without ambiguity between being true or false, provable or refutable. A formal system is complete if there are no *ignorabimus* (Berto 2008).

Form the syntactic point of view a formal system is (a) *syntactically consistent*, or *coherent*, if it does not allow for proving a formula φ *and* its negation $\neg\varphi$; (b) *syntactically complete* if it gives proof to φ *or* $\neg\varphi$. Therefore, from the syntactic point of view, what matters is the *provability*.

20 Probably the greatest difference between formalism and intutitionism is that the latter allows for excluded middle, statements to be both true and false simultaneously. Thus in modern nonstandard analysis (Robinson 1966) that allows for infinite real numbers and their infinitesimal reciprocals, those infinitesimals are both equal to zero and not equal to zero (Hellman 2008; Rosser 2012).~~.~~

21 While logicism accepts the law of the excluded middle, it does not demand bivalence, allowing for systems of many valued logics such as those based on Boolean algebra(Rosser and Turquette, 1952).

From the semantic point of view a formal system is (c) *semantically consistent*, or *correct*, if, according to a given interpretation, it gives proof *only* to true formuli; (d) *semantically complete* if, according to a given interpretation, it gives proof to *all* the true formuli. Therefore, from the semantic point of view, what matters is *interpretability* of formuli, for which it makes sense asking whether if a formula is true or false.[22]

Coherence is therefore stronger than correctness because the former is merely a syntactic notion while the latter is semantic, hence it takes an interpretation. Moreover, a syntactically inconsistent formal system gives proof to a formula and to its negation at the same time, so realizing a contradiction which, according to a given semantic interpretation, cannot be proved to be true: it gives proof to false statements just like a contradiction tells something false. On the other hand, according to a given interpretation, an incomplete formal system can give proof to false statements even if it is syntactically coherent: in a sense, it tells something false without contradiction. Therefore, one is interested i~~o~~n coherent systems only, this is because in inconsistent formal systems one may deduce whatever one~~she~~ may please.

To explain the relevance of such notions to economics consider that it is a formalized science and that science is that investigation providing sure knowledge about its objects. When speaking the language of mathematics, if science is theoretical its knowledge has the value of truth by syntactic consistency (i.e. coherence), if science is applied its knowledge has the value of truth by semantic consistency (i.e. correctness). The difference is that in the first case it takes a theory to be coherent to provide a valid basis of knowledge, regardless of the outcome of test with facts, in the second case a theory is to be correct, hence and interpretation of reality is needed to take care of the test with facts.

We note more details. The first issue is the imbalance between theoretical and applied economics and, intimately entangled with this, the second issue is that both approaches to economics are considered as valid bases of knowledge even if their specific outcomes do not reciprocally reconcile. To explain the reasons of such an imbalance, and to provide an alternative, it is worth drawing a parallel between economics and physics.

Among many, a difference is that physical facts happen not depending on the opinion of physicists while this is not so~~ as~~ regarding ~~to ~~economists for economic phenomena. As a matter of fact, an economy exists in the real world as a physical phenomenon exists, both influence the lives of everyone, and both can be idealized to be formally treated. But~~, but theoretical physics cannot fail the test with facts while, as said, theoretical economics can. Therefore~~, physics is ~~fully~~

22 The ~~here adopted ~~notions of syntactic consistency, syntactic completeness, semantic consistency and semantic completeness are reported in Berto (2008).

concerned with correctness while economics is more concerned with coherence.[23]

T~~After that, t~~he thought of economists influences institutions and regulators that normatively influence the behavior of citizens, while no physicist may influence nature, other than artificially on purpose ~~o~~for knowledge in experiments.~~ According to the [SL, this principle and the following might be included in the suggested appendix, although these cannot be formalized, at least to the best of my knowledge: *Principle of Verifiability*, for a theory to be a valid (i.e. meaningful) basis of knowledge it is necessary and sufficient that its contents are true as tested by facts. According to the *Principle of Falsifiability*, for a theory to be a valid basis of knowledge its statements should be independently falsifiable (i.e. there should exist some evidence negating a statement in the same conditions it has been asserted) such that if one is found to be false, while being a premise of other statements, then the statement and its consequences are to be refuted.~~

Therefore, if theoretical implications of physics do not match with results of applied physics the theory is then put under revision before being considered as a valid basis of knowledge: what matters is semantic consistency or correctness. On the contrary, if theoretical implications of economics do not match with results coming from applied economics, the economic theory is still considered as a valid basis of knowledge if it is coherent from the logic point of view: what matters is syntactic consistency or coherence.

Related with these, another difference is the following. Theoretical physics is deeply grounded and dialectically related to applied physics' results, for which it seeks to develop a formal description of real phenomena by means of mathematics.~~: both cooperate to develop knowledge with the same dignity~~. Often applied physics drives theoretical physics~~ which, more often than not, inspires applied physics~~. On the contrary, applied economics estimates with some meaningful significance the outcomes of theoretical deductions, yet we regularly see even when the results do not fit the theory this does not seem to undermine the theory noticeably among those who developed it.

Theoretical physics is *developed by means* of mathematics and applied physics involves mathematics to provide measurable instruments~~ement instrumental apparatuses~~, but in the end both planes of the mathematically spoken discourse must meet to pass the test with real facts. Applied economics involves mathematics to provide measurements or estimates, but theoretical economics is *developed upon* the axioms of the mathematical economics formal system. Since the criterion of scientific in economics is not based on semantic consistency (i.e. correctness) but on syntactic

23 We recognize that increasingly parts of physics have come to resemble economics in not being easily tested empirically, this is most seriously the case with string theory, which at the current time remains an untested purely theoretical model competing with various alternatives, with some physicists noting the similarity to the situation in theoretical mathematical economics (Smolin 2006).

consistency (i.e. coherence) it then may happen that applied and theoretical planes of the mathematically spoken discourse do not intersect, surprisingly without consequence because there is no need to pass the test with facts for an axiomatic theory to be a valid basis of knowledge.

As a formally mathematized science, the purpose of economics is to know its objects and the mechanism ruling the economy, not only to know and describe them but also to solve real problems to improve societal organizations and the well-being of citizens. It then seems that the true object of economics is an economy as a real world fact, but this is only a part of the story. Indeed, due to formalization, a great effort is spent on formalizing idealized notions that maintain a weak connection with the real world~~, often on the terminological level only~~.

If semantic consistency and the test with facts are considered then it can be decided whether a theory is correct or wrong. On the contrary, by considering syntactic consistency only, it might happen that a theory is neither true nor false: it is undecidable. Moreover, given that economics is mainly concerned with internal coherence, it may happen that a theory ends up with statements it cannot prove to be false or true according to the axiomatic set of the formal system: two different interpretations might end up with opposite results with neither decidable. In such cases a theory allows for some *ignorabimus* and it is incomplete. This is not a mere matter of ignorance in the sense of a poor information set.

As reported above, completeness might be either syntactic or semantic. From the syntactic point of view a theory is complete upon its set of axioms if its formuli are provable or refutable. From the semantic point of view a theory is complete if all the true formuli are proved, and this requires a notion of truth. After Tarski (1956) the truth cannot be defined as an absolute but only in a finalized manner, thus it comes clear that interpretation is essential. For instance, Keynesian and Monetarist are competing interpretations. All the theories based on such interpretations provide coherent descriptions and deductions about economic facts, not infrequently with theorems. Nevertheless, such theoretical proposals have some kind of *ignorabimus*. Hence many economic theories may be incomplete, including GET.

~~To clarify this, economic theories are often over-simplified by the Friedman (1953) instrumentalist "as if" principle combined with the reductionist paradigm. Both allow to focus the attention on the main stylized facts to be explained in terms of the economic behavior of ideal types of agents. After that, and in advance of considering any kind of formalism, at the roots there are economic axioms about rationality of agents, preferences, information, efficiency of markets and so on.~~

All such aspects are translated from the ordinary language of economics thought to the artificial language of mathematics, whose classical form has it own axioms.[24] Hence, the axiomatic set of mathematical economics increases and statements that~~which~~ can be express~~pronounce~~ed in the ordinary language of economics can be expressed~~become pronounceable~~ in the artificial language of mathematical economics. This clearly improves the capability of investigation but it also shifts the focus of attention from the substance of the economic thought to the forms needed for its development. Thus, statements have a truth-value once written as theorems, the constitutive elements of theories.

To state theorems following a sequence of demonstrations based upon the axiom set of a formal system requires syntactical consistency. Since there can be different interpretations, the semantic consistency is only relative to single (often competing) theories, and since the principles of verifiability and falsifiability do not apply it turns out semantic completeness is impossible and that all that matters for a theory to be a valid basis of knowledge is its internal coherence only. As we shall see this opens the door to the Gödel-Rosser's theorems. Therefore, an economic axiomatic theory is not only semantically incomplete for the reasons explained above but it is also syntactically incomplete due to its syntactic consistency.

From the formalistic point of view, for any mathematized science, incompleteness of a theory is not an expression of our ignorance, or the non-exhaustiveness of the information set, or of the set of functions relating different sorts of information, or of the set of primitive entities that~~which~~ can be isolated to specify a theory.[25] The incompleteness we are concerned with is that of the formal system: a theory is incomplete if its axiom set is incomplete. Indeed, if formally operating by means of correct reasoning, from a decidable set of coherent axioms one cannot deduce false statements and, demonstrations can conclude with theorems, it can also end up with some not provable although truthful statement.

~~If the formal system upon which the theory is specified is not powerful enough to prove all and only the statements constituting a theory, then it may happen that some not provable but truthful statements can be deduced. If this happens then the theory is incomplete because its set of coherent axioms is not complete. Thus using correct reasoning, from coherent premises one will deduce not only theorems but also some truthful conjectures that cannot be proved.~~

24 The standard set is usually called ZFC for “Zermelo-Frankel-Choice” axioms (Kleene 1967). Competing forms of mathematics may not assume all those axioms, as is the case with intuitionistic mathematics already mentioned.

25 Consider having isolated all and only the primitive entities we are interested in, for instance prices and action plans in any given place and time, all the functions appropriated to represent exchange preferences and, moreover, being able to effectively compute the values of all the functions at any value of their arguments. Then consider the representations of the general economic equilibrium according to the specification of Debreu: a finite number of goods and a finite number of individuals, each choosing the best action plan in terms of production, if a firm, or consumption, if a household, all defined in a convex and closed space-time on the multi-dimensional field of real numbers. The representations may be exhaustive.

The incompleteness we are concerned with is that of the formal system of mathematical economics of the general equilibrium theory following from Arrow and Debreu (1954) and mainly, due to its explicit axiomatic approach, that following from Debreu (1959). The axiomatic development of such theory according to the formalistic school of mathematics is what intimately relates it to the issue of coherence and incompleteness and, by consequence, to the *Incompleteness Theorem*, which is the combined reading of the two Gödel's[26] statements together with the generalization of the first one due to Rosser (1936):[27]

Theorem 1 [GR1] *Any consistent formal system, within which a certain amount of elementary arithmetic can be carried out, is incomplete: i.e. there are statements of the language of the formal system which can neither be proved nor disproved in the formal system.*

Theorem 2 [G2] *For any consistent formal system, within which a certain amount of elementary arithmetic can be carried out, the consistency of the formal system cannot be proved in the formal system itself.*

The *Incompleteness Theorem* ended Hilbert's program while putting a limit to any theory developed upon is axiomatic formal system, provided that it is capable of carrying out at least the Peano Arithmetic. The main implication of this theorem is that of putting a limit to mathematics and mathematically formalized theories: coherence of an axiomatic theory implies its incompleteness, and any attempt to prove coherence by means of the same axiomatic set is doomed to fail because self-reference, or recursion, arises so that one will always shift the limit of the proof a step forward without the possibility of ending up with a definitive solution.[28]

26 The proposed statements consider "consistent" as synonymous with "coherent" and they are retrieved from the Stanford Encyclopedia of Philosophy at https://stanford.library.sydney.edu.au/archives/spr2014/entries/goedel-incompleteness (retrieved in February 2017). ~~In preparation of reading of these theorems it might be useful consulting the biographic note about Kurt Gödel at https://plato.stanford.edu/entries/goedel (retrieved in February 2017). Among many, F~~for a rigorous logic-formal explanation of such theorems the interested reader is addressed to Smorynski (1977) and Beklemishev (2010).

27 In Gödel's original version he was unable to show that simple consistency implied incompleteness within a sufficiently effective system but had to assume the specific ω-consistency. Rosser's "trick" was to use the "Rosser sentence" that "If this sentence is provable, there is a shorter proof of its negation." This allowed for showing the neat connection between simple consistency and incompleteness, despite the apparently more complicated form of the sentence. The combined theorem is more widely known in Europe as the "Gödel-Rosser Theorem" than in the United States, even though discussions there are based on its results, which also more readily link to the discussions regarding computability raised by Turing in connection with the incompleteness theorems.

28 A classic example of this is the Cretan Liars Paradox: "All Cretans are liars, and I am a Cretan," with Gödel's own version of this informally being "This sentence is not provable." Such neverending self-referencing loops are tied to the halting problem of computer science that is at the heart of the problem of incomputability as posed by Turing (1936). This is the deep link between computational complexity and the incompleteness theorems, and Kleene (1967) provided a Turing machine-based proof of Rosser's version of the incompleteness theorem.

Let us now discuss to what extent such a theorem is relevant to economics. As Ingrao and Israel (1991) discussed with a great deal of historical details, among different economic theories that of general equilibrium is the most pervasive. Born in the mind of Walras (1874), the Walrasian interpretation has lastingly conditioned the analysis of those fundamental principles leading to the notion of equilibrium. Since the works of Arrow and Debreu (1954) and Debreu (1959) this stream of literature has grown but the research program following from Walras has not been completed. Nevertheless, the work in the field of general equilibrium theory has not yet come to an end (Gintis 2007; Gintis and Mandel 2012; Mandel et. al. 2013) even after the Sonnenschein–Mantel–Debreu (SMD)[29] theorems and understanding the stringency of conditions for uniqueness in the form of gross substitutability (Debreu, 1970) and that stability depends further on the specific nature of the system's dynamics (Scarf 1960).[30]

T~~Most part of t~~he endless story of~~in~~ general economic equilibrium is due to the axiomatic foundation of the theory. According to Solow (1985) the main flaw is that axiomatization which should have been the fundamental guarantee of the result, and this also extends to ~~the~~ recent DSGE theory.

The main reason ~~o~~for this can be found in the ~~logic~~ formal logic genetics of mathematical economics discussed above. If the axiomatic system at the root of the economics formal system is unable to represent the phenomenology of real economic systems, as coherent as it is, its incompleteness may allow for deducing formal propositions which are not provable nor refutable, as well as it may allow for theorems which do not match with facts. This can be read in~~at~~ the light of *Incompleteness Theorem*: an axiomatic formal theory which is at the same time coherent and complete cannot exist. One must accept incompleteness in pragmatic terms of ignorance provided that theoretical deductions allow for a satisfactory description of "the world as we see it". Since the degree of satisfaction for description is always subjective and arbitrary, it then does not really matter how far such a description is from "the world as it is": hence, ~~according to the "as if" principle,~~ what matters is that a theory is powerful enough to provide coherent deductions.

T~~herefore, t~~he main problem is not the inability to accurately describe facts with all the details one might want. The problem is that the mainstream approach ~~self-justifies and~~ accredits itself of being scientific~~ only~~ due to its internal coherence, regardless of its ~~logic~~ formal logic

29 The combined reading of results due to Sonnenschein (1972, 1973), Mantel (1974) and Debreu (1974) is known as the *SDM Theorem*, see also Arrow and Intrilligator (1982), Kirman (1989) and Rizvi (2006). As formulated by Debreu the SMD Theorem states that for a continuous, homogeneous of degree zero function, with Walras' Law holding, there exists an economy with at least as many agents as goods for which the function is an aggregate demand function with prices bounded away from zero. In short, aggregate demand based on microfoundations is highly arbitrary and variable, only loosely bounded in form.

30 For a comprehensive exposition of all such topics the reader is addressed to Arrow and Intrilligator (1982).

incompleteness. N~~Indeed, by n~~eglecting this aspect, one can ~~is allowed to~~ introduce as many axioms (or *epicycles*) as one~~she~~ might please till a (presumed) proof is obtained of what is to be proved by a theory.[31] I~~And here the *Incompleteness Theorem* explicates its relevance: to contrast incompleteness as pragmatic ignorance of a theory, and in order to give proof of what is to be proved to put forth a theory as scientific because logically consistent, i~~f augmenting the axiomatic set at the root of the formal system by adding new axioms seems enough, one forgets that ~~logic~~ formal logic incompleteness teaches us that there is no possibility to give proof, from the inside, that such an augmented axiomatics is still consistent. I~~Hence, i~~f a theory would be accredited as scientific from the formal ~~and~~ logic point of view concerning its internal coherence, ~~it must be accepted that~~ such a consistency is to be paid at the price of incompleteness, which means that one should accept the idea that undecidable statements can be deduced. ~~, i.e. statements we cannot prove nor disprove.~~ This is because, according to the *First Gödel-Rosser Theorem*, there is no consistent formal system which is also complete and, according to the *Second Gödel Theorem*, no consistent formal system can give proof of its consistency from the inside.

Arising directly from the *Incompleteness Theorem* and its link with the problem of non-computability as formulated by Turing (1936) with the halting problem arising from self-referencing do-loops that never stop, it should be unsurprising that this has come to be understood as a widespread problem for much of economic theory. Most equilibrium models possess serious constraints on being computable, if they are computable at all. Probably the first to pose this for game theory equilibria was Michael Rabin (1957) who showed that "There are games in which the player who can always win cannot do so in practice because it is impossible to supply him with effective instructions regarding how he should play in order to win." While this is an uncomputability result, it looks like it comes straight from the *Second Gödel Theorem*. A more general result would be established by Tsuji, da Costa, and Doria (1998) showing a more general non-computability of Nash equilibria, while Koppl and Rosser (2002) would show how the self-referncing problem leads to non-computability of best response functions in game theory.

The problem of self-referencing and ubiquitous incompleteness with non-computability was posed by Peter Albin (1982), who noted the link between this and the problems associated with aggregation in the capital theory debates. Arrow's student, Alain Lewis (1992) would more directly show limits to computability of general equilibrium using Turing's work, which reputedly influenced Arrow's move to support the establishment of the Santa Fe Institute and to support research on economic complexity (Mirowski 2002). A more general result on non-computability of

[31] Thus despite its repeated empirical failures (Phuong, et al., 2016), more axioms are added to the flawed DSGE theory, much as epicycles were relentlessly added to the failed Ptolemaic theory.

Walrasian equilibria is due to Richter and Wong (1999), with this clearly encompassing such simplifications of GET as the DSGE approach, even though obviously approximate solutions are computed for such models regularly.[32]

4 The Incompleteness Theorem Trap for the General Equilibrium Theory

Coherence of the formal system is inalienable and, mainly in economics, formalization makes coherence a truth-value for axiomatic formal theories. A theory is coherent if it is able to infer truthful deductions without contradiction, even though some of them could be not provable by incompleteness. Therefore, by empowering with new axioms the already existing formal system one feels like being able to prove what was not provable, at least in syntactic terms. But only resorting to the augmented formal system we cannot prove its coherence. Not only, as coherent as it is, with the augmented formal system there could be new truthful conjectures as unprovable statements. Therefore, coherence and incompleteness are entangled and both put a limit to the formal system involved in the theory, and to the theory itself. Such a limit can be overcome only by means of a higher order one, or by means of external statements regarding empirically testable hypotheses.

Thus applicability is a kind of limit. When formalization implies an over-simplification,[33] it could happen that the implications of the axiomatic formal theory do not match facts in the real world, from which the theory takes inspiration. Therefore we have theorems to formally state proved truths, but the formal apparatus is disconnected from the phenomenon, and this determines a weakness in terms of applicability of the theory. As a syntactic mathematical structure, the axiomatic formal theory is emptied of any semantic interpretation and content, which determines the hiatus between theory and practice. Nevertheless, even without empirical evidence the value of truth of the axiomatic formal theory is not undermined, this is because what matters is coherence and not correctness.

If the theory -based on its formal system *FS* with axiomatic set ***AX***- deduces some truthful

32 A broader effort to establish a school of thought on “computable economics” that studies the limits of economic theory based on computability issues arising from the *Incompleteness Theorem* and issues associated with Turing machines has been due to Velupillai especially, who provides excellent broad overviews (Velupillai 2000, 2009). However, it should be noted that a part of his program is to try to reconstruct economic theory based on non-classical mathematical foundations using constructivist approaches such as intuitionism that assume fewer of the classical axioms than do the works cited here that also find non-computability limits profoundly associated with various models of economic equilibria. While we view this research program with interest (Rosser 2012), we do not pursue its implications for this discussion further here.

33 Over-simplification means that the general axiomatic conditions, useful to prove a given statement or to solve a certain problem (e.g. the existence of equilibrium), may need to be “restricted” to further simplify the world in order to prove a new statement or to solve a new problem (e.g. concerning the structure of the equilibrium set) which cannot be proved or solved within the original axiomatic set. In a sense over-simplification may lead to the formal construction of an artificial world where new properties hold while being too far from being realistic or applicable.

statement γ upon which a mathematical model can be specified, it may be that to prove the conjecture γ it takes introducing stronger restrictions in the model. The stronger the restrictions the lower the explanatory power of the theory and the lower the applicability of the model: “If the model that has been specified requires strong assumptions to guarantee the existence of an equilibrium price vector, the explanatory power of the model will be low. In order to evaluate the model, a basic question must, therefore, be answered in the form of axioms that make it possible to prove an existence theorem”.34 Therefore, to evaluate the validity of the conjecture γ, the model's restrictions must be considered as new axioms $\boldsymbol{A}_1$ in the theory. Hence $\boldsymbol{AX}$ becomes $\boldsymbol{AX}_1=\boldsymbol{AX}+\boldsymbol{A}_1$.

So $\boldsymbol{AX}_1$ may prove what the theory cannot prove by means of $\boldsymbol{AX}$ only. But, according to *Second Gödel Theorem*, FS_1 cannot be proved to be coherent from the inside. However, even if no contradiction is found in the theory then according to *First Gödel-Rosser Theorem*, it follows that FS_1 is incomplete again and now the conjecture γ receives a proof to assume the status of theorem but a new conjecture γ_1 can be deduced. Thus the old problem comes again, and again because the axiomatic formal system approach gets entrapped in the *Incompleteness Theorem* each time one tries to solve newly arising problems.

Let us then consider that the axiomatic formal system of economics is that of the GET following from Debreu (1959) and let *FS* be the formal system of interest equipped with Debreu’s axiomatic set $\boldsymbol{AX}$. In order to involve the *Incompleteness Theorem FS* is to be powerful enough to carry out some elementary arithmetic such as the Peano Arithmetic. It is possible, at least in principle algorithmically, to arithmetize GET according to a gödelization procedure. By gödelizing each symbol, statement and demonstration, in principle one can speak of the GET with natural numbers. Therefore, it is possible to apply the *Incompleteness Theorem* to Debreu (1959).

Hence, by formally developing GET as an axiomatic coherent theory upon its formal system, the *Incompleteness Theorem* exerts its limiting effects on Debreu (1959). Thus it follows that from *FS* some truthful but unprovable statements can be deduced; hence the GET of Debreu is incomplete. This is confirmed by several studies[35] that had been forced to introduce more “simplifying hypotheses or assumptions” for the equilibrium to be, at least, locally unique and stable. Thus the *FS* of Debreu proving the existence of the general equilibrium is not powerful enough to prove its uniqueness nor stability.[36] Along with pragmatic limits such empowering axioms are so restrictive of the microeconomic foundation that the model is too unrealistic to be

34 Quotation of Debreu reported in Bryant (2010).
35 See chapters XI and XII of Ingrao and Israel (1991) for a narrative and historical review.
36 As Ingrao and Israel (1991) report, Hildebrand said that Debreu considered that the axiomatic development of the existence theory was not sufficient for a theory of~~n~~ uniqueness and stability. Debreu was obviously concerned with these issues but not from the global point of view, rather he understood that uniqueness and stability could be entangled as local properties of an equilibrium set.

applicable.

~~Therefore, since the *FS* is coherent but incomplete, one can therefore think of introducing new axioms $\boldsymbol{A}_1$ in $\boldsymbol{AX}$ of *FS* to prove what *FS* cannot prove or, in terms of Gödel's speech in 1930, to solve the problem which cannot be solved, e.g. uniqueness of the equilibrium. However, one may wonder if FS_1 is as coherent as *FS*.~~

The only way to prove uniqueness is to substitute its global property with local ones, such that a discrete set of equilibria can be found: with more restrictions it becomes ~~countably~~ finite.[37] It was long known that in a two good pure exchange model gross substitutability was sufficient, but this failed to hold with more goods, and for strict uniqueness under the index theorem such strong global assumptions as that there be only one consumer with a strictly linear input-output technology might be required (Kehoe 1985), with this laying the groundwork for the narrowness of subsequent DSGE modeling.[38]

Furthermore, *a-priori* coherence cannot be guaranteed, indeed one should at least~~-~~ prove that with the new axioms $\boldsymbol{A}_1$ no statement in FS_1 contradicts another one. However, due to the *Second Gödel Theorem*, we know that the proof of coherence cannot be given by solely resorting to FS_1. Obviously, the *Second Gödel Theorem* does not imply that FS_1 is incoherent, it only states that coherence of FS_1 cannot be proved by means of FS_1, while it takes something else outside of FS_1. Therefore, with FS_1 the equilibrium might be proved to be unique while not proving that it exists in the same way it has been proved with *FS*. Hence, to proceed with FS_1 one has new restrictions about existence. Therefore, due to *First Gödel-Rosser Theorem*, we will find new not provable statements in the augmented system, maybe this time a γ_2 concerning equilibrium stability. Clearly, along with this line, the theory will never end due to recursiveness and the entanglement between coherence and incompleteness.

The previous example is a different look at the so called *SMD Theorem* which puts a mathematical limit to the GET as the *Incompleteness Theorem* puts a limit to mathematics, hence to the mathematical economics of the GET too. As Kirman (2004) explains “there is no hope of general result on stability since the 'only' conditions on the aggregate excess demand function that can be derived from strengthening the assumptions on individual preferences” are the standard ones (e.g. the aggregate excess demand must be continuous, it must obey the Walras' Law, it must be homogenous of degree zero and if prices converge to zero excess demand should explode) and they

37 Dierker (1974) showed that if they are transversal~~under mild conditions~~ the number of equilibria is odd.
38 In game theoretic models the conditions can be far more stringent, with Maskin and Riley (2003) spending over two decades to work these out for sealed-bid auctions. Of course in the simple two good Walrasian model the gross substitutability condition also provides stability, but since the famous three good example of Scarf (1960), it has been known that uniqueness does not guarantee stability for more than two goods for such models.

do not guarantee stability. Moreover, since the *SMD Theorem* holds true then the whole GET is limited on the pragmatic level too because, as Kirman writes, equilibria are of interest if they can be attained though a reasonable adjustment process, but it is now known that equilibrium is neither unique nor stable in general and it has been proved that to have some reasonable outcome with an adjustment process, which is more powerful than the "tatonnement", it takes an unreasonable amount of information: hence, no economy is expected to converge to an equilibrium. And if we consider that uniqueness cannot be global but, at best~~least~~, a local property, then multiple equilibria cannot be managed with "comparative statics", so no one knows what is the current equilibrium, what was the previous equilibrium, and what the next one might~~would~~ be.

The difficulty, in this case, is that in the Debreu's interpretation of the GET time does exist but it is hidden in the mathematical structure of the action plans, so the adjustment one seeks can be any but surely it must be an instantaneous one, which is almost unrealistic but, at the formal level of the discourse, this is irrelevant. This is due to the specific topological structure introduced since Arrow and Debreu (1954), and more explicitly described in Debreu (1959), the action plans are such that space (e.g. locations) and time (e.g. transactions dates) are indexing goods, even the same good "here" today and "there" tomorrow are two different goods in the economy. T~~Under suitable further restrictions of the axiomatic set regarding the microeconomic foundation of the original GET, it can be proved that the equilibrium set is discrete and countably finite, hence uniqueness can be introduced as a local property. However, once the problem of existence and local uniqueness of the equilibrium is solved, that of t~~he adjustment process cannot be solved because it makes no sense in a static framework. T~~herefore, t~~o manage the issue of stability of any existent local equilibrium, the axiomatic formal system of the GET must~~is to~~ be augmented ~~and modified~~ to allow for ~~involving~~ dynamics explicitly with explicit intertemporality, and this is what the DSGE approach tries to do~~es~~. ~~However, to mange the issue of equilibrium and dynamics, in terms of intertemporality,~~ O~~o~~ne is forced to reduce the full heterogeneity of the GET by means of reductionistic representative agent hypothesis~~. Therefore, as it will be explained below, reducing heterogeneity to allow for explicit dynamics~~ and assume rational expectations, thu~~mean~~s adding new axioms about the economic behavior of agents while preserving some of the GET's axioms about optimization. Hence, the DSGE approach is subject to~~still provided an axiomatic theoretical framework which undergoes~~ the *Incompleteness Theorem*.

Therefore, if we proceed by successive *epycicles* of axioms in a formal system we will recursively remain entrapped by the *Incompleteness Theorem*. At each *epycicle* the formal system will meet unprovable conjectures which, to be proved, need adding new axioms to reach a satisfactory level of completeness while compromising coherence, but while incompleteness is acceptable coherence is inalienable. Here we find the original sin of axiomatic formal economics: an assumption is not an axiom, the first need of being tested somehow while the second works as a revealed truth.

For a pure formalist (as was Debreu) the axiomatic system might be arbitrary. ~~and this sublimates the “as if” principle of Friedman (1953) with a big concern. Indeed, o~~Once the theory has been arithmetized there exists an algorithm to test whether ~~if~~ a statement is an axiom or not.~~: unfortunately, the standard way of neither put assumptions to test nor it involves such a kind of algorithmic proof. And this creates a problem in terms of logic weakness: a~~Adding new ~~and new~~ axioms by necessity means the formal system can be empowered but the *Second Gödel Theorem* asserts that it cannot be proved to be coherent from the inside, hence the *First Gödel-Rosser Theorem* ~~activates to~~ asserts that unprovable conjectures will be found, and one can even coherently find not truthful statements with incompleteness.

For finding a way out of this coherence-incompleteness trap for DSGE modeling the following provides one possible way. The basic idea is that the *Incompleteness Theorem* has effects on both the GET and the DSGE, since both are axiomatic-formal theories sharing axioms on utility and ~~the~~ optimization. From GET we can reach ~~the~~ DSGE by explicitly introducing time dynamics in the GET, as if we had the same structure of the GET to be intertemporally iterated and calling this a DGET (Dynamic GET). By removing full heterogeneity from the DGET while adding rational expectations then we could end up with a DSGE structure.

We have four main categories:[39] heterogeneity (**H**), optimization (**O**), intertemporality (**I**) and rational expectations (**RE**).

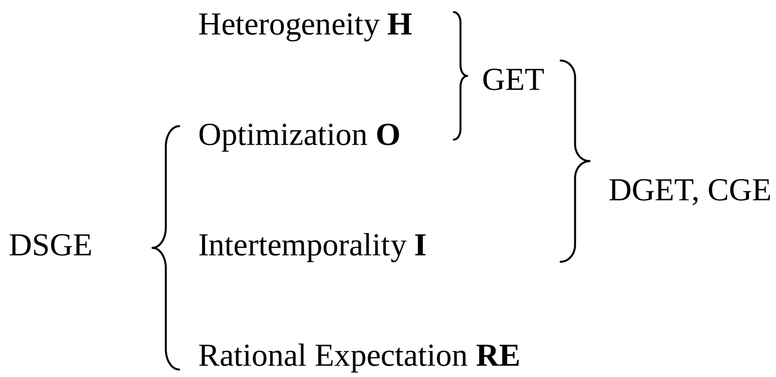

Fig.1 The GET-DGET/CGE-DSGE interpretation scheme.

The Arrow-Debreu general equilibrium theory (GET) involves **H** and **O**. **H** means a lot of agents (consumers and producers) and goods/services indexed by space (locations) and time (selling

39 This set of categories has been discussed with Federico Giri, the scheme GET-DGET/CGE-DSGE if of the authors.

and purchasing dates) as differentiated commodities, hence time is implicitly hidden in the topological structure adopted to solve the problem of equilibrium existence which is a static theory, therefore the problem of stability boils down as meaningless while that of uniqueness persists. **O** means maximizing utility (for consumers) and profit (for producers).

Debreu proved that the equilibrium set is discrete and ~~countably~~ finite, and this solved the problem of uniqueness: in a small neighborhood of an equilibrium point there are no other supply-demand compatibility states that~~which~~ can be considered as equilibrium points, hence equilibrium is locally unique. Not only, being the equilibrium set a subset of the action plans space it must be convex and compact, that means it is limited and closed.

According to the goods-space-time topological structure the GET is a static theory but, if we remove time from the topological structure as indexing goods we can introduce it as an explicit dimension to iterate from *t* to *t+1* the whole GET recursively, this basically means that we can do Computational General Equilibrium (CGE) in an intertemporal framework, as well as the ACE or ABM, although these latter two do not depend on optimization, **O**, in the way the others do. Differently said, removing the hidden time from the GET to make it explicit in an intertemporal framework we can define a DGET involving **H**, **O** and **I**.

If in the DGET we substitute **H** with few representative agents on the supply and demand sides to allow for **RE** we then end up with the DSGE framework~~,~~ that is the DSGE is a DGET which removes **H** and involves **RE**[40]

Therefore, GET – DGET – DSGE share only optimization **O**, which asserts axioms about agents' behavior. But the DSGE still undergoes the *Incompleteness Theorem* because it is axiomatic formal and coherent, hence incomplete, as regarding **O**, **I** and **RE**.

5 How the Axiomatic Formal Theories can Escape from the Coherence-Incompleteness Trap?

Developing an axiomatic formal theory is like playing chess against the *Incompleteness Theorem* and the most reasonable way to try to win is doing our very best not to lose. As in chess game our opponent may move a piece for enticement, the *Incompleteness Theorem* may seduce us in terms of coherence while hiding the winning move of incompleteness, which stimulates us to introduce new axioms so compromising coherence: if this happens the *Incompleteness Theorem* gives us the check mate. Therefore, not to be entrapped in the coherence-incompleteness recursion the most reasonable

[40] We note that there have been efforts to introduce heterogeneous agents into DSGE modeling. Curiously this often takes the form of assuming a continuum of agents a la Aumann in an interval that varies on some characteristic (Krusell and Smith, 1998). This leads to similar dynamics of a representative agent model, with the interval effectively acting as a single agent without any actual interactions between the agents.

move we can play is not to push~~insist~~ beyond our limits with new and new axioms but to open the already existing axiomatic system to reality.

Not to lose the game, we should introduce effective hypotheses which can be tested, case by case: whenever it is possible nothing that~~which~~ can be proved is to be assumed without proof. This is not a general strategy to always win but generic tactics to prove what is to be proved. If the theory turns out to be inconsistent or not matching with facts the theory is to be revised. But if no inconsistency is found after having introduced the previously tested hypothesis, then coherence is preserved and, above all, the implications of the theory will be pretty close to facts.

The advantage of this strategy is evident: we will end up with theoretical proposals whose implications are in agreement with data and we also have a theory to explain the reasons why something happens when something else happens. Of course, we would never have a "general" theory but a "generic" theory which works fine in all the cases that are consistent with the empirically tested hypotheses, those in accord with "the world as it is". Clearly this is a price no formalist is prone to pay, it is not a matter of faith but a matter of method: opening the axiomatic system to testable hypotheses means losing something in terms of theoretical apparatus but, at the same time, means gaining something else in terms of applicability of the theory.

So the question becomes more generally how to move beyond the formalistic limits of general equilibrium theory while still engaging in serious and substantive quantitative modeling of economic systems? We advocate the use of modeling methods that do not rely on optimization that requires axioms that place the analyst in the situation that has been presented here. Agent-based models (ABMs) are one such approach, where no overall equilibrium is necessary, even if some such models do lead to such an outcome. ABMs can embody behavioral assumptions for the agents that are empirically sound as argued above. But these behavioral assumptions do not need to be derived from axiomatic foundations. Rather they reflect real world estimated data. An important source of such data may well be experiments from either lab or field settings, preferably from both with the each confirming the findings from the other.

A behaviorally based ABM approach also opens the door to a greater possibility of modeling evolutionary dynamics of a system.[41] The lack of imposing an equilibrium outcome on the system allows for changes to emerge from the lower level interactions of the agents in the model, hopefully based again on sound empirical findings. It is reasonable that in moving beyond the axiomatic formalism of a GET-DSGE framework, approaches that emphasize the possibility of ongoing

41 The contrast between evolutionary and equilibrium approaches was at the foundation of Veblen's (1898) critique of neoclassical equilibrium theory and his proposal to replace it with the study of the evolution of economic institutions.

change in an evolutionary manner should be made possible. Rather than all-knowing and optimizing agents bound to obey problematic axioms, boundedly rational agents attempting to improve their situation in interaction with the other agents around them can provide a stronger foundation and framework for understanding real economic dynamics in a complex world economy.[42]

6 Concluding Remarks

It has long been a driving dream of many fields of intellectual endeavor to achieve a universal or general theory that explains all things within that field of thought. This effort had its first run in mathematics with the formalism program formulated by David Hilbert at the beginning of the twentieth century. This effort came to a crashing halt with the proof by Kurt Gödel that all sufficiently formal systems that are consistent are not complete, although the full linking of these clearly was made later by J. Barkley Rosser, which is why the theorems involved are sometimes called the Gödel-Rosser theorems.

The effort to develop a universal formalism on axiomatic foundations migrated to economics, especially in the work of Gérard Debreu, who was trained by the hyper-formalistic Bourbaki School of mathematics in France. General equilibrium theory (GET) was the centerpiece of this effort, especially the proof of the existence of Walrasian general equilibrium based on a parsimonious set of foundational axioms. However, certain problems remained unsolved by this effort, notably the questions of uniqueness and stability of such equilibria. While conditions for local uniqueness have been largely established, stability remains a more complicated problem. In any case, to solve in a formal manner involves invoking additional axioms, which places the theorist in the conundrum of facing conflicts over consistency and completeness. These issues extend to the dynamic~~s~~ stochastic case of general equilibrium (DSGE) models as well.

Alternatives to this formalistic conundrum may involve modeling that does not rely on general equilibrium or overall optimization, even if agents may be assumed to be attempting to improve their situations and outcomes. Agent-based modeling (ABMs) offer such an alternative, especially when behavioral assumptions regarding agents are based on sound empirical data, possibly obtained from experiments. Such approaches may be better at modeling the evolutionary dynamics that economies exhibit over time better than the limited possibilities inherent in the

42 See Rosser and Rosser (2015) for further discussion of how behavioral economics derived from Simon (1957) bounded rationality approach aids in understanding complex economic dynamics.

general equilibrium approach.

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