Physicians of JMU health center check up patients during the day at the rate of one every 15 minutes. Assume the check up time is exponentially distributed. It is observed that on the average one patient arrives at the health center in every 20 minutes according to Poisson distribution to join the single waiting line. JMU would like to study the current service level at its health center using M/M/s model in Q.xls. Assume only one physician is doing the check – ups and M/M/1 Queuing model is used for Questions 1 to 7.

Mean check up (service) time $\frac{1}{\mu} = 15$ min/patient Exponentially dist.
mean service rate $\mu = 60$ min/15 min $= 4$ patients/hr.

interarrival
mean arrival time $\frac{1}{\lambda} = 20$ min/patient
mean arrival rate $\lambda = 60$ min/20 min $= 3$ patients/hr
poisson dist.

M/M/s Queuing Model
- s no. of servers
- Exponentially distributed service time
- poisson distributed arrivals

Given $\lambda = 3$ patients/hr,
What is the mean arrival rate λ in 30 minutes?
$\lambda = 1.5$ patients/30 min
What is the mean interarrival time in this case?
mean interarrival time $= \frac{1}{\lambda} = 30$ min/1.5 patients
$= 20$ min/patient

Terminologies:

L_q :	$L_q = \lambda W_q$	mean no of units in queue waiting
W_q :		mean waiting time in queue before service
L :	$L = \lambda W$	mean no. of units in system (queue + server)
W :		mean amt of time in System (queue + service)
S :		no. of servers

P_W : probability of having to wait upon arrival
P_0 : prob. of empty system P_n : prob. of n units in system.

1. What is the probability that more than three and less than six patients arrive at the health center in two hours?

$P(3 < X < 6 \mid \lambda = 6 \text{ in 2 hrs})$

Since $\lambda = 3$ patients in 1 hr

$\lambda = 6$ patients in 2 hrs or $\lambda = 6$ patients/2hrs

poisson prob. dist. for the no. of arrivals in 2hrs.

$$P(X = x \mid \lambda = \ ) = \frac{\lambda^x e^{-\lambda}}{x!} \quad \text{for } x = 0, 1, 2, \cdots,$$

0	1	2	3	4	5	6	7	⋯ ⋯

$P(3 < X < 6) = P(X=4) + P(X=5) = P(X \le 5) - P(X \le 3) = .2945$

$\lambda = 6$ patients/2hrs

$= \text{POISSON}(5, 6, \text{TRUE}) - \text{POISSON}(3, 6, \text{TRUE})$

$= \frac{6^5 * 2.71828^{-6}}{5!} = .16062$

$= \frac{6^4 * 2.71828^{-6}}{4!} = .13385$

What is the prob. that No customer arrives in 30 min?

λ in 30 min $= 1.5$ patients/30min.

$$P(X = 0 \mid \lambda = 1.5 \text{ patients/30min}) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{1.5^0 e^{-1.5}}{0!} = e^{-1.5} = .22313$$

$= \text{POISSON}(0, 1.5, \text{False})$

$= \text{EXP}(-1.5)$

What is the probability that more than three patients arrive in 20 minutes? $\lambda = 1$ patient in 20 min

0	1	2	3	4	5	⋯ ⋯

$P(X > 3 \text{ if } \lambda = 1/20\text{min}) = 1 - P(X \le 3 \text{ if } \lambda = 1/20\text{min}) = \text{POISSON}($

$= P(X=0) + P(X=1) + P(X=2) + P(X=3)) = 1 - \text{POISSON}(3, 1, \text{TRUE}) = 0.01899$

2. Recall that the function for the Poisson distribution has the form = POISSON(x, mean, cumulative). Entering the formula = POISSON(2, 2, TRUE) in Excel will return a value of 0.6767. Show the standard equation with the result for the problem and briefly interpret the meaning of it.

POISSON(x, λ, TRUE) = P(X ≤ x if λ)

POISSON(x, λ, False) = P(X = x | λ) = P(X = x if λ)

└ False for x = a, TRUE for x ≤ a

└ arrival rate λ in the same time duration of x

└ poisson dist. random arrival in time duration

= POISSON(2, 2, TRUE) = P(X ≤ 2 if λ = 2 patients in 40 min)

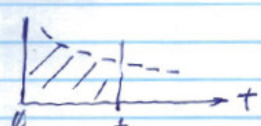
= .6767 because λ = 3 patients/hr

probability that no more than two patients arrive in 40 min.

3. What is the probability that a patient will spend more than 10 minutes for the check up?

probability dist. for exponential service time:

$P(T \le t \text{ if } \mu) = 1 - e^{-t\mu}$



$P(T > t \text{ if } \mu) = 1 - P(T \le t \text{ if } \mu) = 1 - (1 - e^{-t\mu}) = e^{-t\mu}$

$P(T > 10 \text{ min if } \mu = 4 \text{ patients/hr}) =$

$= P(T > \frac{10}{60} \text{ hr if } \mu = 4 \text{ patients/hr})$

$= 1 - P(T \le \frac{1}{6} \text{ hr}) = 1 - (1 - e^{-\frac{1}{6}4}) = 1 - 1 + e^{-\frac{1}{6}4} = e^{-2/3} = .5134$

4. What is the Excel@ formula or function to be used to compute the probability that a patient will spend more than 10 minutes for the check up?

=1−EXPONDIST(1/6, 4, TRUE)

≤ 10 min or 1/6 hr

$\mu = 4/hr$

10 min = 1/6 hr

$1-e^{-t\mu}$

What is the probability that a patient is checked up in exactly 15 min?

$P(T = 15 \text{ if } \mu) = \phi$

t is continuous

What is the probability that a patient is checked up in less than 10 min?

$P(T < 10 \text{ min if } \mu = 4/hr) = 1 - e^{-\frac{10}{60} 4} = 1 - e^{-2/3} = .4866$

=EXPONDIST(1/6, 4, TRUE)

What is the probability that a patient is checked up between 10 min and 20 min?

$P(10\,min < T < 20\,min \text{ if } \mu = 4/hr)$

$= P(T < 20\,min) - P(T < 10\,min)$

$= P(T < \frac{20}{60}\,hr) - P(T < \frac{10}{60}\,hr)$

$= 1 - e^{-4 \cdot 2/6} - (1 - e^{-4 \cdot 1/6})$

$= 1 - e^{-4/3} - 1 + e^{-2/3} = e^{-2/3} - e^{-4/3} = .2498$

=EXPONDIST(2/6, 4, TRUE) − EXPONDIST(1/6, 4, TRUE)

What is the percentage of time that physicians busy checking up patients?

server utilization

$\rho = u = \lambda/\mu = 3/4 = .75 = 75\%$

What is the average number of patients waiting to be checked?

$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{3^2}{4(4-3)} = \frac{9}{4} = 2.25$ patients waiting

What is the average amount of time that a patient spent waiting?

$L_q = \lambda W_q$

$W_q = L_q/\lambda = 2.25/3 = 0.75\,hrs = 45\,min$

5. What is the probability that there is more than one patient in the health center for check -ups?

prob. of more than one patient in the health center for check up!

$n > 1$ in system is ≥ 2 in system, and for $s=1$,

$P(n \geq 2) = 1 - P(n \leq 1) = 1 - (P_0 + P_1) = 1 - \frac{1}{4} - \frac{3}{16} = 9/16$

$P_1 = P_0(\frac{\lambda}{\mu}) = (\frac{1}{4})(\frac{3}{4}) = \frac{3}{16}$

$P_0 = 1 - \frac{\lambda}{\mu} = 1 - 3/4 = 1/4$

6. What is the expected number of patients in the health center?

What is the average number of patients in the physicians office?
(in queue waiting plus the one being checked)

$L = L_q + \lambda/\mu = \frac{\lambda}{\mu - \lambda} = 2.25 + 3/4 = 3 = \frac{3}{4-3}$

What is the ~~average number of patients~~ amount of time a patient spent in the clinic (waiting plus being checked)?

$W = L/\lambda = \frac{1}{\mu - \lambda} = \frac{1}{4-3} = 1 \text{ hr} = 3/3$

7. Suppose the physician receives pay of $100 per hour and the cost of a patient waiting is estimated to be $1 per minute, what is the total cost per hour for JMU to operate its health center?

Given: physician's pay = $100/hr
penalty cost of patient waiting = $1/min

What is the Total hourly cost to operate the health center?

Total cost = $100 * 1 + $1/min * 60 min * 3 = $280/hr

L = 3

One hr

penalty cost/min

S = 1 physician

hourly pay for physician

Assume all of the information is given in Table 1:

- Two or more physician are doing the check – ups and M/M/s model is used for Questions 8 to 10
- All of time units are in hours and
- The arrival rate λ and the service rate μ for M/M/s model may be different than that of M/M/1 model

	Option 1	Option 2	Option 3	Option 4	Option 5
Arrival rate	5	5	15	15	15
Service rate	8	8	8	8	8
Number of servers	**1**	**2**	**2**	**3**	**4**
Utilization	62.50%	31.25%	93.75%	62.50%	46.88%
P(0), probability that the system is empty	0.375	0.5238	0.0323	0.1322	0.1492
Lq, expected queue length	1.0417	0.0676	13.6089	0.6457	0.1276
L, expected number in system	1.6667	0.6926	15.4839	2.5207	2.0026
Wq, expected time in queue	0.2083	0.0135	0.9073	0.043	0.0085
W, expected total time in system	0.3333	0.1385	1.0323	0.168	0.1335
Probability that a customer waits	0.625	0.1488	0.9073	0.3874	0.1447
P1	0.2344	0.3274	0.0606	0.2479	0.2798
P2	0.1465	0.1023	0.0568	0.2324	0.2623
P3	0.0916	0.0320	0.0532	0.1452	0.1639
P4	0.0572	0.0100	0.0499	0.0908	0.0768
P5	0.0358	0.0031	0.0468	0.0567	0.0360

8. Assume there are 15 patients arrive at the health center per hour.
   a. If JMU wants the expected time for a patient to wait in line before being checked to be no more than 3 minutes, which option(s) should JMU take and why?
   b. With three physicians do check – ups, what is the probability that more than one patient is waiting in line?

8. Assume there are 15 patients arrive at the health center per hour.
    a. If JMU wants the expected time for a patient to wait in line before being checked to be no more than 3 minutes, which option(s) should JMU take and why?

M/M/s Queue model using Table on pp. 4

8: Assume $\lambda = 15$ patients/hr.

8.a If JMU expects $W_q \leq 3$ min, which option to take?

or $W_q \leq 3/60$ hr $= .05$ hr

with 15 patients/hr $= \lambda$,

	option 3	option 4	option 5
W_q in min	54.438 min	2.58 min	0.51 min
W_q in hr	.9073 hr	.043 hr	.0085 hr

Either option 4 with $W_q = 2.58$ min $= 0.043$ hr or

or option 5 with $W_q = .51$ min $= .0085$ hr $\leq W_q = 3$ min

8. Assume there are 15 patients arrive at the health center per hour.
    b. With three physicians do check – ups, what is the probability that more than one patient is waiting in line?

$s = 3$, what is probability more than one patient waiting in line?

$n > 1$ in queue is $n \geq 2$ in queue

is $n \geq 2 + (s = 3)$ or $n \geq 5$ in system.

$$p(n \geq 5) = 1 - p(n \leq 4) = 1 - (P_0 + P_1 + P_2 + P_3 + P_4)$$
$$= 1 - (.1322 + .2479 + .2324 + .1452 + .0908)$$
$$= 1 - .8485 = .1515$$

9. if JMU wants to minimize the hourly total operating cost with the arrival rate λ of 15 patients per hour, which option should the bank take and why?

9. For $\lambda = 15$ patients/hr, which option has min hourly cost?

	option 3	option 4	option 5
s =	2	3	4
physician Cost \$100/hr	\$200	\$300	\$400
L =	15.4839	2.5207	2.0026
waiting Cost = \$60/hr	\$929.034	\$151.242	\$120.156
Total Cost/hr	\$1,129.034	\$451.242	\$520.156

Thus, the option 4 with s = 3 should be used with total cost = \$451.242.

10. (13 points) As in the table for the option 4, $P_0 = 0.1322$ for s = 3 (three physicians do the check – ups). Use the M/M/s queuing equation $P_0 = \left[\sum_{n=0}^{s-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^s}{s!}\left(\frac{s\mu}{s\mu-\lambda}\right)\right]^{-1}$ to confirm the value of 0.1332.

For option 4. $P_0 = .1332$ for $s = 3$ physicians doing check ups.

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} + \frac{(\lambda/\mu)^s}{s!}\left(\frac{s\mu}{s\mu-\lambda}\right)\right]^{-1} = \frac{1}{4.6328 + 1.0986 + 2.6667}$$

$$\frac{3 \times 8}{3 \times 8 - 15} = \frac{24}{24-15} = \frac{24}{9} = \frac{8}{3} = 2.6667$$

$$\frac{(15/8)^3}{3!} = \frac{1}{6}\left(\frac{15}{8}\right)^3 = 1.0986$$

$$\frac{\left(\frac{15}{8}\right)^0}{0!} + \frac{(15/8)^1}{1!} + \frac{(15/8)^2}{2!} = 1 + \frac{15}{8} + \frac{1}{2}\left(\frac{15}{8}\right)^2 = 4.6328$$

$$P_0 = \frac{1}{7.5625} = .1322$$