Mean checkup (service) time $\frac{1}{\mu} = 15$ min/patient Exponentially dist.
mean service rate $\mu = 60$ min/15 min = 4 patients/hr.

interarrival
mean arrival time $\frac{1}{\lambda} = 20$ min/patient
mean arrival rate $\lambda = 60$ min/20 min = 3 patients/hr.

Poisson dist.

M/M/3 Queuing Model

$L$ = no. of servers
$\lambda$ = Exponentially distributed service time
$\mu$ = Poisson distributed arrivals

Given $\lambda = 3$ patients/hr.

What is the mean arrival rate $\lambda$ in 30 minutes?
$\lambda = 1.5$ patients/30 min.

What is the mean interarrival time in this case?
mean interarrival time $= \frac{1}{\lambda} = 30$ min/1.5 patients
$= 20$ min/patient

Terminologies:
$L_q$ : $L_q = \lambda W_q$ mean no. of units in queue waiting
$W_q$ : mean waiting time in queue before service
$L$ : $L = \lambda W$ mean no. of units in system (queue + server)
$W$ : mean amount of time in system (queue + server)
$S$ : No. of servers
$P_{n}$ : Probability of leaving to wait upon arrival
$P_0$ : prob. of empty system
$P_n$ : prob. of n units in system
1. \( P(3 \leq X < 6 | \lambda = 6 \text{ in 2 hrs}) \)

Since \( \lambda = 3 \) patients in 1 hr
\[ \lambda = 6 \text{ patients in 2 hrs or } \lambda = 6 \text{ patients/2hrs} \]

Poisson prob. dist. for the no. of arrivals in 2 hrs:
\[
P(X = x | \lambda = \lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \text{ for } x = 0, 1, 2, \ldots
\]

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | \ldots
|---|---|---|---|---|---|---|---|---

\[
P(3 \leq X < 6) = P(X = 4) + P(X = 5) = P(X \leq 5) - P(X \leq 3) = 0.2945
\]

\[\lambda = 6 \text{ patients/2hrs}\]

\[
= \text{poisson}(5, 6, \text{TRUE}) - \text{poisson}(3, 6, \text{TRUE})
\]

\[
= \frac{6^5 e^{-\frac{6}{2}}} {5!} = 0.16062
\]

\[
= \frac{6^4 e^{-\frac{6}{2}}} {4!} = 0.13385
\]

What is the prob. that no customer arrives in 30 min?

\[ \lambda \text{ in 30 min} = 1.5 \text{ patients/30 min.} \]

\[
P(X = 0 | \lambda = 1.5 \text{ patients/30 min}) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{1.5^0 e^{-1.5}}{0!} = e^{-1.5} = 0.22313
\]

\[L = \text{poisson}(0, 1.5, \text{False})\]

What is the probability that more than three patients arrive in 20 minutes?

\[ \lambda = 1 \text{ patient in 20 min} \]

| 0 | 1 | 2 | 3 | 4 | 5 | \ldots
|---|---|---|---|---|---|---

\[
P(X > 3 | \lambda = 1 \text{ per min}) = 1 - P(X \leq 3 \text{ if } \lambda = 1 \text{ per min}) = \text{poisson}(\lambda = 1, 3, \text{TRUE})
\]

\[= p(x = 0) + p(x = 1) + p(x = 2) + p(x = 3) = 1 - \text{poisson}(3, 1, \text{TRUE}) = 0.01999\]
2. \[ \text{poisson}(X, \lambda, \text{TRUE}) = P(X \leq x \mid \lambda) = P(X = x \mid \lambda) = P(X = x \mid \lambda) \]

- False for \( x = 0 \), True for \( x \leq 0 \)

arrival rate \( \lambda \) in the same time duration of \( X \)

- poisson dist. random arrival in time duration

\[ \text{poisson}(2, 2, \text{TRUE}) = P(X \leq 2 \mid \lambda = 2 \text{ patients in 40 min}) \]

= .6767

because \( \lambda = 3 \text{ patients/hr} \)

- probability that no more than two patients arrive in 40 min.

3. probability dist. for exponential service time:

\[ P(T \leq t \mid \mu) = 1 - e^{-t \mu} \]

\[ P(T > t \mid \mu) = 1 - P(T \leq t \mid \mu) = 1 - (1 - e^{-t \mu}) = e^{-t \mu} \]

- \( P(T \geq \frac{10}{60} \text{ hr} \mid \mu = 4 \text{ patients/hr}) = 1 - P(T \leq \frac{10}{60} \text{ hr} \mid \mu = 4 \text{ patients/hr}) = 1 - (1 - e^{-\frac{10}{60} \mu}) = 1 - 1 + e^{-\frac{1}{3}} = e^{-\frac{1}{3}} = .5134 \]

- \( 1 - \text{expdist}(x, 4, \text{TRUE}) \)

\[ L \leq 10 \text{ min or } \frac{1}{6} \text{ hr} \]

\[ \frac{L}{\mu} = 4 / \text{hr} \]

\[ 10 \text{ min} = \frac{1}{6} \text{ hr} \]

- \( 1 - e^{-t \mu} \)

What is the probability that a patient is checked up in exactly 15 min?

\[ P(T = 15 \mid \mu) = \phi \]

\( \phi \) is continuous.
What is the probability that a patient is checked up in less than 10 min?

\[ P(T < 10 \text{ min} \mid \lambda = 4/\text{hr}) = 1 - e^{-\frac{10}{60} \times 4} = 1 - e^{-\frac{2}{3}} = 0.4866 \]

\[ = \text{Expondist}(\frac{1}{6}, 4, \text{TRUE}) \]

What is the probability that a patient is checked up between 10 min and 20 min?

\[ P(10 \text{ min} < T < 20 \text{ min} \mid \lambda = 4/\text{hr}) \]
\[ = P(T < 20 \text{ min}) - P(T < 10 \text{ min}) \]
\[ = P(T < \frac{20}{60} \text{ hr}) - P(T < \frac{10}{60} \text{ hr}) \]
\[ = 1 - e^{-\frac{2}{3}} - (1 - e^{-\frac{1}{3}}) \]
\[ = 1 - e^{-\frac{2}{3}} - 1 + e^{-\frac{1}{3}} = e^{-\frac{2}{3}} - e^{-\frac{1}{3}} = 0.24982 \]

\[ = \text{Expondist}(\frac{1}{6}, 4, \text{TRUE}) - \text{Expondist}(\frac{1}{6}, 4, \text{TRUE}) \]
What is the percentage of time that physicians are busy checking up patients?

\[ P = \frac{\mu}{\lambda} = \frac{3}{4} = 0.75 = 75\% \]

What is the average number of patients waiting to be checked?

\[ L_q = \frac{\lambda^2}{2\mu(\mu - \lambda)} = \frac{3^2}{2 \cdot 4 \cdot (4 - 3)} = \frac{9}{4} = 2.25 \text{ patients waiting} \]

What is the average amount of time that a patient spent waiting?

\[ L_q = \lambda W_q \]
\[ W_q = \frac{L_q}{\lambda} = 2.25 \cdot \frac{1}{3} = 0.75 \text{ hrs} = 45 \text{ min} \]

6. What is the average number of patients in the physician's office?

\[ L = L_q + \frac{\lambda}{\mu} = \frac{9}{4} + \frac{3}{4} = 3 = \frac{3}{4 - 3} \]

What is the average amount of time a patient spent in the clinic (waiting plus being checked)?

\[ W = \frac{1}{\mu - \lambda} = \frac{1}{4 - 3} = 1 \text{ hr} = \frac{3}{3} \]
5. The probability of more than one patient in the health center for check-up is $n > 1$ in system is $\geq 2$ in system, and for $s=1$:

$$p(n \geq 2) = 1 - p(n \leq 1) = 1 - (q_0 + q_1) = 1 - \frac{1}{4} - \frac{3}{16} = \frac{9}{16}$$

$$p_1 = p_s(\frac{2}{4}) = \frac{1}{4} \frac{3}{4} = \frac{3}{16}$$

$$p_0 = 1 - \frac{a}{\mu} = 1 - \frac{3}{4} = \frac{1}{4}$$

7. Given: physician's pay = $100/hr
penalty cost of patient waiting = $1/min

*What is the total hourly cost to operate the health center?*

Total cost = $100/1 + $1/min * 60 min * 3 = $280/hr

- $L = 3$
  - One hr
  - Penalty cost/min
- $S = 1$ physician
  - Hourly pay for physician
M/M/3 Queue Model using Table on pp. 4

8. Assume λ = 15 patients/hr.

8.a. If Simu expects Wq ≤ 3 min, which options to take?

Or Wq = \frac{3}{60} hr = 0.05 hr

With 15 patients/hr = λ,

\begin{align*}
\text{Option 3} & : Wq \text{ in min} = 54.438 \text{ min} \\
\text{Option 4} & : Wq \text{ in min} = 2.58 \text{ min} \\
\text{Option 5} & : Wq \text{ in min} = 0.51 \text{ min}
\end{align*}

\begin{align*}
Wq \text{ in hr} & : 0.9073 \text{ hr} \\
& : 0.043 \text{ hr} \\
& : 0.0085 \text{ hr}
\end{align*}

Either option 4 with Wq = 2.58 min = 0.043 hr or

or option 5 with Wq = 0.51 min = 0.0085 hr ≤ Wq = 3 min.

8.b. S = 3, what is probability more than one patient waiting in line?

n ≥ 1 in queue is n ≤ 2 in queue

\[ p(n ≥ 5) = 1 - p(n ≤ 4) = 1 - (p_0 + p_1 + p_2 + p_3 + p_4) \]

\[ = 1 - (0.1322 + 0.2479 + 0.2324 + 0.1452 + 0.0908) \]

\[ = 1 - 0.8485 = 0.1515 \]
9. For $\lambda = 15$ patients/hr, which option has min hourly cost?

<table>
<thead>
<tr>
<th>Option</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S =$</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>Physician Cost $/100/hr</td>
<td>$200</td>
<td>$300</td>
<td>$400</td>
</tr>
<tr>
<td>$L =$</td>
<td>15.4839</td>
<td>2.5207</td>
<td>2.0026</td>
</tr>
<tr>
<td>Waiting Cost $/hr</td>
<td>$929.034</td>
<td>$151.242</td>
<td>$120.156</td>
</tr>
<tr>
<td>Total Cost $/hr</td>
<td>$1129.034</td>
<td>$451.242</td>
<td>$520.156</td>
</tr>
</tbody>
</table>

Thus, the option 4 with $S = 3$ should be used with total cost $= \$$451.242.

10. For option 4, $P_0 = .1332$ for $S = 3$ physicians doing checkups.

\[
P_0 = \left[ \sum_{n=0}^{S-1} \frac{(\lambda/n)!}{n!} \left( \frac{S \mu}{S \mu - \lambda} \right)^n \right] + \frac{(\lambda/n)!}{S!} \left( \frac{S \mu}{S \mu - \lambda} \right)^S = \frac{1}{4.6328 + 1.0986 + 2.6667}
\]

\[
\frac{3 + 8}{3 + 8 - 15} = \frac{24}{24 - 15} = \frac{8}{9} = \frac{2}{3} = 0.6667
\]

\[
\frac{(15/8)^3}{3!} = \frac{1}{6} \left( \frac{15}{8} \right)^3 = 1.0988
\]

\[
\frac{15}{8} \cdot \frac{15/8}{1!} + \frac{(15/8)^2}{2!} = 1 + \frac{15}{8} + \frac{1}{2} \left( \frac{15}{8} \right)^2 = 4.6328
\]

\[
P_0 = \frac{1}{7.5625} = .1332
\]
M/G/1 Queue

M/M/s Queue with finite queue positions