Solution to ILP problem:

Rounding LP relaxation leads to a solution

- feasible
- not feasible

Branch-and-Bound

Stopping rules

Excel© solver for ILP problem

Integer Linear Programming

Blue Ridge Hot Tubs.

$$\begin{array}{llllll}
\text{Max}: & 350X_1 + 300X_2 & & & & \text{profit} \\
\text{S.T.} & 9X_1 & + & 6X_2 & \leq 1{,}566 & (\text{Labor}) \\
 & 12X_1 & + & 16X_2 & \leq 2880 & (\text{Tubing}) \\
 & 1X_1 & + & 1X_2 & \leq 200 & (\text{pump}) \\
 & X_1, & & X_2 & \geq 0 & \\
 & X_1, & & X_2 & \text{must be integers} &
\end{array}$$

LP Relaxation of ILP Problem

optimal solution of the LP relaxation

Bounds for Objective function Value (OFV) of ILP

OFV of LP relaxation is Upper bound for max problem (ILP)

OFV of LP relaxation is Lower bound for min problem (ILP)

Conclusion: ILP optimal OFV cannot be better than its LP relaxation.

ILP Example: Employee Scheduling Problem

Shift	Days off	Wage	Day of Week	Workers Req.
1	Sun. & Mon	\$655 + \$25	Sun.	18
2	Mon & Tue	\$655 + \$50	Mon.	27
3	Tue & Wed	\$655 + \$50	Tue.	22
4	Wed & Thur	\$655 + \$50	Wed.	26
5	Thurs & Fri	\$655 + \$50	Thur.	25
6	Fri & Sat	\$655 + \$25	Fri.	21
7	Sat & Sun	\$655	Sat.	19

Decision Variables:

SM = # of workers assigned to shift 1 with Sun & Mon. off.
MT = ... 2 .. Mon & Tue —
TW = ... 3 .. Tue & Wed —
WT = ... 4 .. Wed & Thur —
TF = ... 5 .. Thur & Fri —
FS = ... 6 .. Fri & Sat —
SS = ... 7 .. Sat & Sun —

OBJ. MIN: $680SM + 705MT + 705TW + 705WT + 705TF + 680FS + 655SS$

Constraints: LHS RHS

on Sun:	$0SM$	$+1MT$	$+1TW$	$+1WT$	$+1TF$	$+1FS$	$+0SS$	≥ 18
on Mon:	0	0	1	1	1	1	1	≥ 27
on Tue:	1	0	0	1	1	1	1	≥ 22
on Wed:	$1SM$	$+1MT$	$+0TW$	$+0WT$	$+1TF$	$+1FS$	$+1SS$	≥ 26
on Thur:	1	1	1	0	0	1	1	≥ 25
on Fri:	1	1	1	1	0	0	1	≥ 21
on Sat:	1	1	1	1	1	0	0	≥ 19

$SM, MT, TW, WT, TF, FS, SS \geq \phi$ and must be integer

A Capital Budgeting Problem

Proj.	($1000) Expected NPV	Capital (in $1000) Required in Year 1	Year 2	Year 3	Year 4	Year 5	($1000) Funds
1	$141	75	25	20	15	10	~~$250~~
2	$187	90	35	0	0	30	~~$75~~
3	$121	60	15	15	15	15	~~$50~~
4	$83	30	20	10	5	5	~~$50~~
5	$265	100	25	20	20	20	~~$50~~
6	$127	50	20	10	30	40	
Funds ($1000)		$250	$75	$50	$50	$50	

Decision Variable:

$$P_i = \begin{cases} 1, & \text{if Project } i \text{ is selected} \\ 0, & \text{otherwise} \end{cases} \qquad i = 1, 2, 3, 4, 5, 6$$

OBJ: Max: $141 P_1 + 187 P_2 + 121 P_3 + 83 P_4 + 265 P_5 + 127 P_6$

						LHS	RHS	
S.T. $75 P_1$	$+ 90 P_2$	$+ 60 P_3$	$+ 30 P_4$	$+ 100 P_5$	$+ 50 P_6$		$\le \$250$	(Y1)
$25 P_1$	$+ 35 P_2$	$+ 15 P_3$	$+ 20 P_4$	$+ 25 P_5$	$+ 20 P_6$		$\le \$75$	(Y2)
20	0	15	10	20	10		$\le \$50$	(Y3)
15	0	15	5	20	30		$\le \$50$	(Y4)
10	30	15	5	20	40		$\le \$50$	(Y5)

$P_1, P_2, P_3, P_4, P_5, P_6$ must be binary

Additional possible constraints:

$P_1 + P_3 + P_6 \le 1$ no more than one

$P_1 + P_3 + P_6 = 1$ exact one

$P_4 - P_5 \le 0$

P_4	P_5	feasible	
0	0	yes	none
1	1	yes	Both
0	1	yes	P_5 only
1	0	No	